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Dynamic and topological interplay in adaptive networks

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Theory and applications of complex networks pervade nonlinear sciences. This chapter is devoted to adaptive networks which combine topological evolution of the network with dynamics in the network nodes – a property which yields a rich dynamical interplay between the state and the topology of the network. Recently, the dynamics of adaptive networks has been investigated in a number of parallel studies from different fields, ranging from genomics to game theory. Here we review these recent developments and show that they can be viewed from a unique angle. We demonstrate that all these studies are characterized by common themes – most prominently: complex dynamics and robust topological self-organization based on simple local rules. With our review we aim to provide an overview of these central properties of adaptive networks, to outline the different perspectives from which they have been considered and finally to set an agenda for future research.

1.1

Introduction

Complex networks are ubiquitous in nature and are gaining increasing attention in the nonlinear sciences. They occur in a large variety of real-world systems ranging from ecology and epidemiology to neuroscience, socio-economics and computer science [1, 53, 54]. Important examples include ecological food webs, the network of social contacts, the internet, the road network and the neuron network in our brain. While physics has for a long time been concerned with well-mixed systems, lattices and spatially explicit models, the investigation of complex networks has in the recent years received a rapidly increasing amount of attention. In particular, the need to protect or optimize natural networks as well as the quest for creating robust

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and efficient technical nets that exploit similar organizing principles prove to be strong incentives for research.

Beside the identification and characterization of network structure in real natural systems, most recent studies revolve around two key questions: what are the topological properties of a network that is evolving in time and, secondly, how does the functioning of the network depend on these properties? These questions have given rise to two distinct lines of research. The first of these is concerned with the *dynamics of networks*. Here, emphasis is put on the structure of the network, which itself is regarded as a dynamical system that grows or changes over time according to specific, often local, rules. Notable examples include the investigation of the small-world property of social networks [78] and the formation of scale free topology in growing networks, like citation networks [62] or the internet [4]. These and a large number of subsequent works have revealed that simple evolution rules, such as preferential attachment or selective rewiring, can be used to generate complex network topologies. Many of these rules are not just useful theoretical algorithms, but mimic natural processes of network formation.

The second major line of network research has focused on the *dynamics on networks*. Here, the network represents an ensemble of dynamical systems, where each node is attributed a dynamic state and the interaction between individual units is described by the adjacency matrix of the network. Thus, the topology of the network remains static but the states of the nodes change according to local evolution rules. Important processes that are studied within this framework include synchronization in ensembles of coupled oscillators [5] or contact processes, such as opinion formation and epidemic spreading [7, 43, 51, 52, 60]. These studies have made it clear that the network topology can have a strong impact on the dynamics of the nodes. For instance it was shown that vaccination of a fraction of the nodes can not stop epidemics on a scale free network [51, 60].

Until recently, the two lines of network research described above were pursued almost independently in the physical literature. While there was certainly a strong interaction and cross-fertilization, a given model would either describe the *dynamics of* a certain network or the *dynamics on* a certain network. Nevertheless, it is clear that in most real world networks the evolution of the topology is invariably linked to the state of the network and vice versa. Consider for instance a road network. The topology of the network, that is the pattern of roads, influences the dynamic state, i.e. the flow and density of traffic. But, if traffic congestions are common on a given road, it is likely that new roads will be build in order to decrease the load on the congested one. In this way a feedback loop is formed in which the topology of the network affects the dynamics on the network, while the dynamics on the network has an influence on the time evolution of the topology. This feedback loop can

give rise to a complicated mutual interaction between a time varying network topology and the nodes' dynamics. Networks which exhibit such a feedback loop are called *coevolutionary* or *adaptive* networks [28]. More examples of this class of networks are discussed below.

Based on the successes of the two lines of research mentioned earlier, it is the next logical step to bring these strands back together and to investigate the dynamics of adaptive networks which combine the time evolution of the topology with that of the state of the nodes. Indeed, a number of papers on the dynamics of adaptive networks have recently appeared. Since adaptive networks occur over a large variety of scientific disciplines they are currently investigated from many different directions. While present studies can only be considered as a first step toward a general theory of adaptive networks, they already crystallize certain general insights. Especially these studies show that the interplay of network state and topology leads to interesting new physical phenomena. Despite the thematic diversification, the reported results, considered together, show that certain dynamical phenomena repeatedly appear in adaptive networks: the formation of complex topologies, robust dynamical self-organization, spontaneous emergence of different classes of nodes from an initially inhomogeneous population, and complex mutual dynamics in state and topology. In the following we argue that the mechanisms that give rise to these phenomena arise from the dynamical interplay between state and topology. They are therefore genuine adaptive network effects that can not be observed in non-adaptive networks.

In this chapter we review the present state of research in the dynamics of adaptive networks. The text is strongly guided by a previous comparative study of adaptive networks across disciplines [28]. It is our aim to provide an overview of the central properties of adaptive networks, to outline the different perspectives from which they have been considered and finally to set an agenda for future research. In particular, we focus on new dynamical effects that are revealed in these previous studies and the methods that are used to study them. We point out that many recent findings reported mainly in the physical literature describe generic dynamical properties of adaptive networks. Because of the ubiquity of adaptive networks these phenomena can potentially be important for a wide range of different applications. Some other models are also included which are not adaptive networks in the narrow sense, but have natural extensions that would turn them to fully adaptive networks. As will become apparent in the following, most adaptive networks that are presently studied have such 'almost adaptive' ancestors. On the other hand certain very applied models which technically contain adaptive networks, but in which the focus is clearly on different questions, are omitted or mentioned only very briefly in this review.

We start in Sec. 1.2 by giving essential definitions for the discussions that follow. Thereafter, in Sec. 1.3 we discuss several examples that illustrate the abundance of adaptive networks in the real world and also in certain classes of applied models. After these two initial sections we proceed to the core of this review. In Sec. 1.4 we discuss adaptive Boolean networks, which illustrate the ability of adaptive networks to self-organize robustly towards criticality. Other less obvious, but no less intriguing forms of the self-organization are discussed in Sec. 1.5 while we review investigations of adaptive coupled map lattices. In particular, it is shown that a spontaneous ‘division of labor’ can be observed in which the nodes differentiate into separate classes, which play distinct functional roles in the network. Further examples of this functional differentiation of nodes are discussed in Sec. 1.6, which focuses on games on adaptive networks. Finally, in Sec. 1.7 we discuss the dynamics of spreading of opinions and diseases on social networks, which shows that the adaptive networks can exhibit complex dynamics and can give rise to new phase transitions. We conclude this review in Sec. 1.8 with a summary, synthesis and outlook.

1.2

Adaptive networks: a definition

Before we start into the discussion of dynamics of adaptive networks let us define the notion of adaptive networks more precisely. For this purpose we first recapitulate some widely used network terminology. Thereafter we introduce some more specific terms relating to the dynamics of networks on which our definition of adaptive networks is based.

1.2.1

Basic definitions of graph theory

Any treatment of complex networks resides on the terminology of graph theory. Here we just present some basic definitions and the most commonly used terminology to provide some basic knowledge. For a more thorough introduction we refer the reader to one the excellent review articles (see e.g., [1,13,53,54]). As usual we define a network as an ensemble of *nodes* (also called *vertices*) which are connected by K (directed or undirected) *links* (or *edges*). The nodes form the principal elements of the network and may represent the basic units of the system under investigation. The total number of nodes is called the *size* of the network and here denoted as N .

The edges of a network usually represent some kind of interaction or relation between the nodes. Together all edges define the network’s *topology*, which denotes a specific pattern of connections between the network nodes.

Two nodes are said to be *neighbors* or *adjacent* if they have a common link. The *neighborhood* of a node corresponds to the set of all adjacent nodes in the graph. The topology can be described by the *adjacency matrix* a_{ij} , where each element $i, j = 0 \dots N$ is taken from $\{0, 1\}$. If two nodes i and j are adjacent, then the adjacency matrix $a_{ij} \neq 0$. Depending on the network under consideration the links can be *directed* or *undirected*. The adjacency matrix of an undirected matrix is symmetric $a_{ij} = a_{ji}$. In general, the links may be of different nature (e.g., inhibiting or activating) and can have different *weights* (interaction strength). This is accordingly described by the weight matrix w_{ij} , where each element is a real number.

The *degree*, k_i , of a node i is the number of nearest neighbors to which it is connected. In a directed network one has to distinguish between the *in-degree*, k_i^{in} , and the *out-degree*, k_i^{out} , corresponding to the number of edges entering or leaving the node. The total degree, then, is the sum $k_i = k_i^{in} + k_i^{out}$. The *mean degree* or *connectivity*, $\langle k \rangle = \frac{1}{N} \sum_i k_i$, is defined as the mean of the individual degrees of all nodes in the network.

An important quantity to characterize a network's topology is its *degree distribution* $P(k)$ which describes the probability that a randomly selected node has a certain number of links. Important examples are the Poisson degree distribution, $P(k) = e^{-\langle k \rangle} k^k / k!$, which is formed by a network in which a fixed number of nodes are randomly connected Erdős-Rényi (*random graph*). The Poisson degree distribution is characterized by a modal hump at the mean degree and exponentially decreasing tails. In contrast, several real-world networks are rather described by power-law degree distributions of the form $P(k) \sim k^{-\alpha}$. Such networks are called *scale-free* and arise for example in a growing network in which new nodes are preferentially connected to nodes which have already many connections (*preferential attachment*). In scale free networks some vertices, the so-called *hubs*, have a degree that is orders of magnitude larger than the average.

Another useful measure to describe the structural and dynamical properties of a network are degree-degree correlations, i.e. correlations between the degree of different nodes on the network. A very natural approach would be to consider the correlations between two adjacent nodes which may be expressed, for example, by the conditional probability $P(k|k')$ that an arbitrary neighbor of a node of degree k has degree k' . Usually, however, it will more easy to compute the average degree k_{nn} of the nearest neighbor of a node of degree k , which is described as $k_{nn} = \sum_{k'} k' P(k|k')$. In the special case that there are no degree correlations, the average degree of the nearest neighbors of a node is independent of its degree k , and k_{nn} is given by $k_{nn} = \langle k^2 \rangle / \langle k \rangle$.

If k_{nn} is an increasing function of the degree k then nodes with a large degree tend to connect to nodes of a large degree. In this case the network is called *assortative*. In the opposite case, if k_{nn} is a decreasing function of the degree

k , nodes with a large degree tend to connect to nodes of a small degree and the network is called *disassortative* [52]. In undirected networks the degree correlation can be computed as the cross correlation $r_{\text{corr}} = \sigma_q^{-2} \sum_{ab} ab(e_{ab} - q_a q_b)$ where $q_a = (a + 1)\rho_{a+1} / \sum_k k\rho_k$, e_{ab} is the probability that a randomly chosen link connects nodes with the degrees $a + 1$ and $b + 1$, and σ_q^2 is the variance of the distribution q_a [52].

Usually, most of the networks of interest are sparse, meaning that only a small fraction of all possible links are present. As a consequence, two randomly chosen nodes i and j of a network will in general not be connected by a direct link. However, it may still be possible that the two non-adjacent nodes are connected through a sequence of l intermediate links. Such a set of links is called a *walk* between nodes i and j of length l [13]. Two nodes are *connected* if there is at least one walk connecting them. A *path* is defined as a walk in which all nodes and links are distinct. Finally, a *loop* or *cycle* is defined as a path starting and terminating in the same node. A path of length three is called a *triple* and a loop of length three is called a *triangle*.

The property of connectedness between two nodes is transitive. If two nodes i and j are connected and also the nodes j and k are connected, then also i and k will be connected. This property can be used to partition a network into non-overlapping equivalence classes of connected nodes, which define the *network components*. Another measure for network transitivity is the clustering, which measures the probability that if node A is adjacent to node B and node B to node C , then also A is adjacent to C , or, in the terminology of social networks, whether the friend of your friend is also your friend. More formally the clustering coefficient C is defined as $C = 3N_{\Delta} / N_3$, where N_{Δ} is the number of triangles and N_3 the number of triples in the network.

Based on the notion of a path one can define several measures of distance on a network. The *shortest path* between two nodes is the path that traverses the minimal number of links between the two nodes. The *closeness* of a node is the length of the mean shortest path to all other nodes in the network. The *diameter* of a graph is the average shortest path length between all nodes in the network. The *betweenness* of a node is the number of shortest path that go through the node. Finally, a network is said to have the *small-world* property if it has a large clustering coefficient, but still most nodes can be reached from the others through a small number of connections, so that the diameter of the network is relative small.

1.2.2

Dynamic and evolving networks

In most cases of interest the nodes of a network have a *dynamic state*. This may be a discrete variable characterizing the node (occupied/non-occupied, infected/susceptible, spin up/down, active/inactive), a scalar variable (such

as a density, concentration, flow etc.), or a more complicated construct like, e.g., a lookup table describing a strategy in a game. Collectively we refer to the state of all nodes as the *state of the network*. Note, that depending on the context in the literature the state of a network is used either to describe the state of the network nodes or the state of the whole network including the states and the topology. In this review we use the term state to refer exclusively to the collective state of the nodes. Thus, the state is a-priori independent of the network topology.

All models considered in this review are *dynamic networks*, in the sense that the state of the nodes changes dynamically in time. These changes can generally be described by a, possibly stochastic, mathematical function, which depends on the current state of the nodes, external parameters and the network topology. In other words, a dynamic network is an ensemble of interacting dynamical systems, where the network connections define the strength and direction of the the interactions.

A node is said to be *frozen* if its state does not change over the long time behavior of the network. In certain systems discussed here the state of a frozen node can change nevertheless on an even longer (topological) time scale.

Furthermore, we distinguish between *static networks*, in which the connections remain fixed in time and *evolving networks*, where the network topology is allowed to change as a function of time. Again, depending on the context, the terms 'dynamics' and 'evolution' are used in the literature to refer to a temporal change of either the state or the topology of a network. In this review we use '*dynamics*' exclusively to refer to a temporal change in the state of a node, while the term '*evolution*' describes temporal changes in the topology.

Depending on the model under consideration, in evolving networks the change of the topology can consist of several possibilities. These are listed in increasing order of complexity (clearly, the later changes give rise to a larger class of topological dynamics than the former):

- Changes in connection weights or the nature of links
This is the weakest form of network evolution. The principal network topology (who connects to whom) remains conserved and only the nature of the links is modified. Such changes include evolutions in the connection weights, i.e. changes from strong to weak links, but could also involve other changes in the nature of links, such as switches from activating to inhibiting links or in the directionality of the connection.
- Rewiring of links
Here the network topology is changed by rewiring of links. There are two possibilities in which a certain link may be reconnected. Either the link is totally removed and replaced by another one, or the link remains attached to one node, but only the second node is changed. Rewiring

keeps the total number of nodes, N , and of links, K , constant.

A famous example of a network that is evolving via rewiring of connections is given by the small world model from Watts-Strogatz [78]. Here, starting from a regular lattice (the pristine world), a certain fraction q of links are randomly rewired. Even if the fraction q is very small, which means that only a small perturbation in the topology has been done and the network locally remains unchanged, it was shown that the global properties, such as mean distances between two nodes, are strongly changed.

- Creation and deletion of links

Links are created or removed (deleted) from the network. This results in a temporal evolution of the topology, but is more general than rewiring. While the (number of) nodes of the network remains unchanged, the number of links K will change over time.

- Creation and deletion of nodes

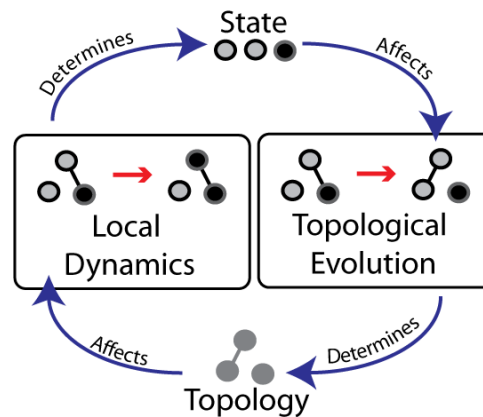
Nodes are created and removed from the network. This is the most drastic form of network evolution. If a node is deleted, some rule is needed that determines the fate of the links that used to connect to the node. Frequently, all the in- and out-going connections of the node are destroyed as well. Or, if a new node is inserted into the network, for this new node, new connections must be placed in. Obviously creation and deletion of nodes goes together with a change of both N and K .

A famous model for an evolving network with increasing number of nodes was presented by Barabasi and Albert [3]. In their model, starting from a small initial network, successively new nodes are inserted. Each new node has m connections, which are attached preferentially to the other nodes j in dependence of their degree k_j . This means that nodes with a large degree have a higher chance to receive additional new links. As was shown in [3] such a rich-get-richer mechanism leads to scale free networks, which are characterized by a power-law degree distribution $P(k) \sim k^{-\alpha}$.

1.2.3

Adaptive networks

With these definitions we are able to describe what we understand as an adaptive network [28]. *Adaptive* networks are evolving, dynamic networks, in which the topology changes in dependence of the dynamic state of the nodes, while the dynamics of the state depends on the topology (see Fig. 1.1). Note, that this definition excludes skew-product networks: dynamic networks in



Tab. 1.1 In adaptive networks the local dynamics in the state of the nodes is interwoven with the topological evolution in the network structure. Thus, the temporal evolution of the topology depends on the dynamics of the nodes, while the dynamics of the nodes is affected by the topology. In this way a feedback loop is created in which a dynamical exchange of information is possible.

which network evolution takes place independently of the state of the nodes or in which the dynamics of the state are independent of the topology.

In almost all real world networks there is some feedback loop that connects the dynamics of the nodes to the network evolution. In this sense almost all real world networks can be considered to be adaptive. However, this does not imply that the adaptive nature of these networks necessarily plays a dominant role in the overall dynamics. This can be understood by considering the involved time scales. For most networks there is a typical dynamic timescale, characterizing the time in which the state of the nodes can change, and a typical evolution time scale over which the network topology changes. If the dynamic timescale is much larger than the evolution time scale, we have the classic evolving network and the dynamic state can be neglected. On the other hand, if the evolution time scale is much larger than the dynamic timescale, then we have practically a fixed network. In contrast to truly adaptive networks, in which the dynamics of topology and state happen approximately on the same timescale, we can expect that the dynamical interplay between state and topology in scale-separated networks is often weak. Therefore, we can define adaptive networks in a strong sense as networks for which these two time scales are close, so that the interaction between these different types of dynamics must be taken into account. However, in the following it will become apparent that in certain scale-separated adaptive networks a dynamical interplay between network state and topology takes place nevertheless.

Finally, in some of the examples discussed here, an adaptive dynamical interplay can only be observed transiently. In these examples the system typi-

cally approaches an attractor on which the network topology stops to evolve in time, while the dynamics of the states can continue. In other systems the dynamical interplay between topology and state continues on the attractor of the system. Note that, although this means that the topology and state never settles down to a static pattern, emergent properties (e.g., mean degree of nodes, degree correlations, number of nodes in a certain state) can approach a steady state. It is therefore useful to distinguish between long-term adaptive networks in which an adaptive interplay persists on the attractor and transient adaptive networks in which an adaptive interplay is only transiently observed.

1.3

Ubiquity of adaptive networks across disciplines

Adaptive networks arise naturally in many different applications. Although studies that target the interplay between network state and topology have only recently begun to appear, models containing adaptive networks have a long tradition in several scientific disciplines. In the introduction we have already mentioned the example of a road network that can be considered as a prototypical adaptive network. Certainly, the same holds for many other technical distribution networks such as power grids [71], the mail network, the internet or wireless communication networks [26,42]. In all these systems a high load on a given component can cause component failures, e.g. traffic jams or electrical line failures, with the potential to cut links or remove nodes from the network. On a longer timescale, high load will be an incentive for the installation of additional connections to relieve this load – thereby giving rise to the above described adaptive interplay, where the state of the network effects the topology, which in turn effects the state.

Essentially the same mechanisms are known to arise in natural and biological distribution networks. Take for example, the vascular system. While the topology in the network of blood vessels directly controls the dynamics of blood flow, the blood flow also exhibits a dynamic feedback on the topology. One such process is arteriogenesis, where new arteries are formed to prevent a dangerous restriction in blood supply (ischemia) in neighboring tissues. This adaptive response in the topology of blood vessels is triggered by a steep pressure gradient that develops along the shortest path within the interconnecting network [69].

More examples of adaptive networks are found in information networks like neural or genetic networks. As will be shown in Sec. 1.4 the functioning of these networks puts relatively tight constraints on the dynamics and topology of the network. In the training of an artificial neuronal network for example it is obvious that the the strength of connections and therefore the topology has

to be modified depending on the state of the nodes. The changed topology then determines the dynamics of the state in the next trial. Also in biological neural and genetic networks some evidence suggests that the evolution of the topology depends on the dynamics of the nodes [34].

In the social sciences networks of relationships between individuals or groups of individuals have been studied for decades. On the one hand important processes like the spreading of rumors, opinions and ideas take place on social networks – and are influenced by the topological properties. On the other hand it is obvious that, say political opinions or religious beliefs, can in turn have an impact on the topology, when for instance conflicting views lead to the breakup of social contacts, while new links are formed preferentially between the likeminded.

In game theory there is a long tradition to study the evolution of cooperation in simple agent based models. In recent years spatial games that are played on a social network have become very popular. While most studies in this area so far focused on static networks, one can easily imagine that the willingness of an agent to cooperate has an impact on his social contacts or business relations. To our knowledge the huge potential of games on adaptive networks and the absence of previous investigations in this area was first pointed out by Skyrms and Pemantle [73].

Games on adaptive network have recently become a hot topic in the engineering literature where they are called network creation games. These are currently investigated in the context of evolutionary engineering [70, and references therein].

Further examples of adaptive networks are found in chemistry and biology. One paradigmatic example is provided by the immune system, in particular the vertebrate immune system, which constitutes a highly parallel, distributed dynamical system and involves large, diverse populations of migratory cells (the human immune network contains about 10 trillion cells). These immune cells are able to communicate in a networked interaction, with the ability for a rapid adaptive response to external stimuli. Thus, more appropriately the immune system is described as an immune network as proposed by N. K. Jerne [38]. The immune network is highly adaptive. It uses learning, memory, and associative retrieval to solve recognition and classification tasks. In particular, it learns to recognize relevant patterns and remember patterns that have been seen previously, for example upon vaccination. Models for the immune network have been proposed already for some time [38,59].

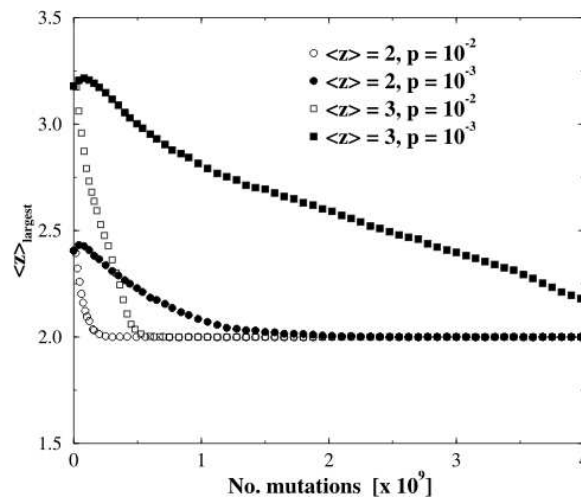
A model of an adaptive chemical network, originally proposed by Jain and Krishna, is studied in [37,72]. In the model the nodes of the networks are chemical species which interact through catalytic reactions. Once the population dynamics has reached an attractor the species with the lowest concentration is replaced by a new species with randomly generated interactions.

Although the topology of the evolving network is not studied in great detail, this model shows that the appearance of a topological feature – an autocatalytic loop–has a strong impact on the dynamics of both state and topology of the network.

While Jain and Krishna focus on the evolution of chemical species, their work is clearly inspired by models of biological evolution. In ecological research models involving adaptive networks have a long tradition. A prominent area in which adaptive networks appear is food web evolution. Food webs describe communities of different populations that interact by predation. A food web can be represented by a directed graph in which the nodes correspond to populations while the edges correspond to predator-prey interactions. In general the state of a node consists at least of the population size, but– depending on the model – may contain additional information about evolutionary traits of the species. In almost all models the abundance of a species, i.e. the dynamic state, depends on the available prey as well as on the predation pressure, both of which depend in turn on the topology of the network. It is very reasonable to assume that Nature does not choose randomly from all possible ecologies, but that instead individual species adapt to their environment so as to enhance their own survival. Many models have attempted to include such adaptation [14,15,18,75]. These models often assume that the population goes extinct if its abundance drops below a critical threshold. In such a case the node is removed from the network, and consequently the dynamics of the topology depends on the state of the network.

The examples discussed above show that adaptive networks appear in a large variety of different contexts. However, the nature and dynamics of the adaptive feedback as such has to-date only been investigated in a relatively small number of studies. In the following sections we focus on papers that specifically investigate the adaptive interplay of state and topology and illustrate the implications this interplay can have.

A prominent ancestor of research in adaptive networks is [12]. In this work Christensen et al. discuss a variant of the famous Bak-Sneppen model of macro-evolution [2]. The model describes the evolution of a number of populations, represented as nodes of a network in which the (undirected) links correspond abstract ecological interactions. The state of each node $i = 1 \dots N$ is a scalar variable, f_i , that denotes the population's evolutionary fitness. Initially this fitness is assigned randomly. Thereafter, the model is updated successively by replacing the population with the lowest fitness by a new species with random fitness. The replacement of a species is assumed to affect also the fitness of the populations it is interacting with. Therefore, the fitness of all neighboring species (that is, species with a direct link to the replaced one) are also set to random values. In the original model of Bak and Sneppen the underlying network is a one dimensional chain with periodic boundary con-



Tab. 1.2 For different initial conditions the connectivity $\langle z \rangle_{\text{largest}}$ of the largest connected cluster of the adaptive boolean network studied by Christensen et al. self organizes towards the critical value of 2. Source: K. Christensen et al. Phys. Rev. Lett. **81** (11), 1998 [12] Fig. 3.)

ditions, so that every population has exactly two neighboring populations. In other words, the degree of each node equals two, $k_i = 2$. It is well known that this model gives rise to avalanches of species replacements which follow a scale-free size distribution [2].

In the paper by Christiansen et al. the simple topology of Bak-Sneppen model is replaced by a random graph [12]. The paper focuses mainly on the evolutionary dynamics on networks with static topology. However, in the second to last paragraph a model variant is studied in which the replacement of a population can affect the local topology. If the replaced population has a lower degree than the species in the neighborhood, there is a small probability that a new link is added that connects to the replaced species. But, if the replaced population has a higher degree than the species in the neighborhood, one link that connects to the replaced species is removed with the same probability. The addition of this evolution rule turns the model into a truly adaptive network and effectively changes the mean degree $\langle k \rangle$. A variant of this model has recently been discussed in greater detail [24]. By numerical simulation Christensen et al. find that the mean degree in the largest cluster of nodes approaches two, $\langle k \rangle_{\text{largest}} \rightarrow 2$ (see Fig. 1.2) – exactly the same mean degree as the linear chain used in the original Bak-Sneppen model. This finding is remarkable since it suggests that adaptive networks are capable of robust self-organization of their topology based on local rules. This observation triggered a number of subsequent studies which will be discussed in the next section.

1.4

Robust self-organization toward criticality in Boolean networks

In order to understand the functioning of adaptive networks it is reasonable to focus on conceptually simple models. In Boolean networks the state of a given node is characterized by a single Boolean variable $\sigma_i(t) = \pm 1$. Therefore Boolean networks with variable topology offer a particularly simple and well-studied framework for the study of dynamical phenomena. Two prominent applications of Boolean networks are neural and gene regulatory nets, in which the state of a given node indicates whether a certain gene is active (i.e., being transcribed) or whether a certain neuron is firing.

It is known that Boolean networks are capable of different types of dynamical behavior, including chaotic and stationary (frozen) dynamics [76]. At the boundary between stationarity and chaos, lies often a narrow transition region, where oscillatory dynamics can be observed and the density of frozen nodes exhibits power-law scaling. According to biological reasoning, neural as well as gene regulatory networks have to be close to or on this 'edge of chaos' to function properly (e.g., to code for different distinct cell types or allow meaningful information processing). A central question is how the networks manage to stay in this narrow parameter region while undergoing topological changes in the course of biological evolution and individual development. It is likely that the adaptive nature of these networks plays a central role in this self-organization towards the critical oscillatory or quasi-periodic states.

One frequently used form of Boolean models to describe regulatory and neural nets are threshold networks. In the simplest form these networks consist of a fixed number of N nodes, each corresponding to a Boolean variable $\sigma_i(t) = \pm 1$, which indicates whether the corresponding node is active or inactive. The nodes are connected by a variable number of directed links with discrete interaction weights $w_{ij} \in \{-1, 0, +1\}$. The updating rule for the dynamics is defined by

$$\sigma_i(t+1) = \text{sgn} \left(\sum_j w_{ij} \sigma_j \right). \quad (1.1)$$

Depending on the topology an active node, $\sigma_i = 1$, exerts an influence on other nodes through its outgoing links. Each of these links can either promote (if $w_{ij} = 1$) or inhibit (if $w_{ij} = -1$) the activity of its direct neighbours in the network. If the inputs received by a node exceed a certain threshold, say if a node receives more promoting than inhibiting signals via its links, the node becomes active; otherwise it is inactive. These rules lead to static or chaotic dynamics or cycles of different length.

In order to study topological self-organization Bornholdt and Rohlf [9] used a Boolean threshold network in conjunction with an update rule for the topol-

ogy. The time evolution of the system is simulated until a dynamical attractor, say a limit cycle, has been reached. At this point the rewiring rule is applied. Rewiring of the local connections of a node depends on the average activity $A_i(t)$ of that node which is defined as the time average in a time interval $[T_1, T_2]$

$$A_i = \frac{1}{T_2 - T_1} \sum_{t=T_1}^{T_2} \sigma_i(t). \quad (1.2)$$

Now, a randomly chosen node is monitored for one period of the attractor or, in case of chaotic dynamics, for a long fixed time. If the state of the node changes at least once during this time, $|A_i| < 1$, it loses a random link. However, if the state remains unchanged for the whole duration, $|A_i| = 1$, a link from a randomly selected node is created. In short, ‘frozen’ or ‘quiet’ nodes grow links, while ‘dynamical’ or ‘active’ nodes loose links.

By numerical simulation Bornholdt and Rohlf show that, independently of the initial state, a certain level of connectivity is approached. If the number of nodes N is changed the emerging connectivity $\langle k \rangle$ follows the power law

$$\langle k \rangle = 2 + cN^{-\delta}, \quad (1.3)$$

with $c = 12.4$ and $\delta = 0.47$. Therefore, in the case of large networks ($N \rightarrow \infty$) self-organization towards the critical connectivity

$$\langle k \rangle_c = 2 \quad (1.4)$$

can be observed. This is explained by further simulations which show that in large networks a topological phase transition takes place at $\langle k \rangle = 2$. In this transition the fraction of ‘frozen’ nodes drops from one to zero. Before the transition all nodes change their state in one period of the attractor, while above the transition almost no node changes its state at all. This means that in a large network, the proposed rewiring algorithm almost always adds links if $\langle k \rangle < 2$, but almost always removes links if $\langle k \rangle > 2$. In this way a robust self-organization towards the dynamically critical state takes place. This form of self-organization is highly robust as it does not depend sensitively on the initial topology or the choice of parameters.

As is pointed out in [9] and later in a different context in [8] these results illustrate an important principle: dynamics on a network can make information about global topological properties locally accessible. In an adaptive network this information can feed back into the local dynamics of the topology. Therefore, the adaptive interplay between the network state and topology can give rise to a highly robust global self-organization based on simple local rules. Note that adding links randomly, as described above, can lead to the formation of, apparently non-local, long distance connections. However, since the

targets of the links are randomly determined no distributed information is used. In this sense topological evolution rules that add or remove random links can be considered as local rules. Further it is worth mentioning that this genuine adaptive network effect can be observed also in networks where topological evolution and dynamics of the states take place on separate time scales, as the example presented in [9] shows.

These findings inspired several subsequent investigations that extended the findings of Bornholdt and Rohlf [8, 10, 11, 39, 46, 48, 64]. A natural generalization is to replace the threshold function Eq. (1.1) by more general Boolean functions. In the Kauffman networks studied in [10, 46, 48] random Boolean functions are used, which are represented by randomly created lookup tables. In [48] these lookup tables are created with a bias p so that a random input leads to activation with probability p and deactivation with probability $1 - p$. In this way networks are created in which the critical connectivity $\langle k \rangle$ can be tuned by changing p ,

$$\langle k_i \rangle = \frac{1}{2p(1-p)}. \quad (1.5)$$

Although a different rewiring rule is used, only allowing for disconnection, self-organization of the system towards the critical state (from above) is still observed.

Similarly, in other studies the authors note that adaptive networks give rise to realistic topologies. For instance Ref. [46] shows that in networks of finite size, the system approaches a broad degree distribution with an average that is slightly above the critical value – a property that is observed in many real world genetic networks.

Another interesting application of adaptive Boolean networks is neutral evolution. In this case only those changes in the topology are allowed, which leave the current attractor of the system unchanged and therefore code for the same cell type. This was studied for example in [10, 11]. In both works it was found that the evolution leads to networks with relatively low connectivity – another property that is shared by real world genetic networks. The low connectivity is explained by the fact that neutral mutations become less frequent with increasing connectivity. Hence the neutral evolvability of a network decreases with increasing complexity. Another interesting result from the investigation of neutral evolution is that the dynamics of the network connectivity exhibits a self-similar punctuated equilibrium behavior [11]. In this equilibrium the lifetime T of a species is distributed as T^{-2} , which agrees well with empirical data from fossil records.

The work described above shows that already very simple adaptive networks can exhibit complex dynamics. In order to find further examples of sets of interesting rules an exhaustive search over a large class of adaptive net-

work models is desirable. Indeed, first attempts in this direction for Boolean networks have been reported in [68]. In particular, a numbering scheme is proposed that allows to enumerate all adaptive networks in a given class. A similar formal, cellular-automaton-inspired approach is used in [74].

Finally let us remark, that beside the mechanism described above there exists an alternative mechanism for making information on the global state locally available, which again can be utilized to robustly self organize the system. This ‘dual’ mechanism applies if the topology of the network changes much faster than the state. For illustration consider the following toy model: In a given network links are established randomly, but links between nodes of different states are instantaneously broken. These rules lead to a configuration in which every node of a given state is connected to all other nodes in the same state. This means that if a given node has, say, five links there are exactly five other nodes in the network that have the same state. Global information on the states has become locally available through the topology. This information can now feed back into the dynamics of the states on a slower timescale.

1.5

Adaptive connection weights in coupled oscillator networks

In the previous section we have discussed the adaptive interplay between state and topology as a dynamical feedback that can drive systems towards criticality. Here, we show that a very similar feedback loop can, in a slightly different setting, guide the self-organization toward different non-trivial topologies.

1.5.1

Leadership and the division of labour

One possible outcome of an adaptive interplay a network’s state and topology is a spontaneous ‘division of labour’: the emergence of distinct classes of nodes from an initially homogeneous population. This phenomenon was first described by Ito and Kaneko [35, 36] in an adaptive network of coupled oscillators. It is remarkable that these authors state with great clarity, that their work was motivated by the new dynamical phenomena that can be expected in adaptive networks.

Ito and Kaneko study a directed, weighted network of coupled logistic maps $f(x) = ax(1 - x)$. Denoting the state of a given map i at time t as x_t^i the temporal dynamics is defined by the set of equations

$$x_{t+1}^i = (1 - c)f(x_t^i) + c \sum_j w_t^{ij} f(x_t^j), \quad (1.6)$$

where $0 < c < 1$ is a parameter that determines the overall strength of the coupling and w_t^{ij} is the time dependent matrix of connection weights which determine the strength of the connections. The connection weights are updated in discrete time steps according to

$$w_{t+1}^{ij} = \frac{[1 + \delta g(x_t^i, x_t^j)]w_t^{ij}}{\sum_j [1 + \delta g(x_t^i, x_t^j)]w_t^{ij}}. \quad (1.7)$$

Here, the parameter δ represents the plasticity of the connection strengths and $g(x_t^i, x_t^j) = 1 - 2|x_t^i - x_t^j|$ is a monotonically decreasing function of the absolute value of the difference between its arguments. This update rule increases the connection strength between oscillators in similar states, while keeping the total incoming connection weight experienced by a single node normalized to one. Due to the normalization the matrix w_t^{ij} in general will be asymmetric.

From the investigation of globally coupled maps it is well known that these systems can be classified into four phases, which characterize the degree of synchronization and clustering among the units. These phases correspond to either coherent, ordered, partially ordered or desynchronized behavior; depending on the parameters a and c [40]. The adaptive network studied by Ito and Kaneko [35,36] does not exhibit a partially ordered phase, since the topological feedback has a strongly stabilizing effect on synchronized clusters. In the coherent and ordered phases the connection weights settle in a fixed pattern, so that the adaptive behavior is only transient. The main focus of the paper is therefore on the desynchronized phase (in the parameter regime $3.7 < a < 4$ and $0 < c < 0.2$), in which a long-term adaptive network is formed. Inside the desynchronized regime Ito and Kaneko identify three regions of different dynamical behavior, which they denote as static, dynamic I and dynamic II. In the static region the nodes form pairs, which recombine only rarely. In the dynamic I region the nodes are connected randomly and rewiring occurs frequently as a result of shifting connection strength. The in- and out-degree distributions are Poissonian, as would be expected for a random graph.

The most interesting behavior is found in the dynamic II region. Here the out-degree distribution becomes bimodal as two distinct classes of nodes are formed that differ by their effective outgoing degree. Even a network in which some nodes are of high degree while other nodes are of low degree could still be considered to be homogeneous on average if every node has a high degree at certain times and a low outgoing degree at others. However, in the model of Ito and Kaneko this is not the case: despite the ongoing rewiring of individual links, a node that has a high/low outgoing degree at some point in time will generally have a high/low outgoing degree also later in time. Note that the outgoing degree indicates the impact that a given node has on the dynamics

of others in the network. In this sense one could describe the findings of Ito and Kaneko as the emergence of a class of ‘leaders’ and a class of ‘followers’. Thus, the networks exhibits a self-organized formation of hierarchies or, to use a more neutral metaphor, a spontaneous ‘division of labour’ in which the nodes differentiate to assume different functional roles.

A similar division of labour was subsequently observed in a number of related systems which can be interpreted as simple models of neural networks [9, 27, 77]. As a common theme, in all these models the topological change arises through a strengthening of connections between elements in a similar state – a rule that is for neural networks well motivated by empirical results [61].

As in the case of the genetic networks discussed in the previous section, the subsequent works showed that simple adaptive models reproduce certain features observed in real world networks. For instance it was shown in [27] that an adaptive neural network approaches intermittent clustering dynamics and organizes itself into a small world topology. This behavior was later shown to be a robust property in networks above a given size by van den Berg and van Leeuwen [77]. Although their model differs from the one used by Ito and Kaneko, van den Berg and van Leeuwen also report the spontaneous division of labor. In this case the two classes that arise correspond to nodes of local importance, forming tight clusters, and nodes of global importance, which connect these clusters.

A common theme that appears many studies of adaptive networks of this type (i.e., strengthening of connections between similar nodes and weakening of connections between different nodes) are wide degree distributions. Even in systems in which no distinct classes of nodes emerge the strengthening of connections between similar nodes often illustrate a strong self-organized heterogeneity among the nodes. A notable example is the formation of a scale free topology reported in [22] and [23].

1.5.2

Self-organization towards synchronizability

From a technical point of view the emergence of strong heterogeneity in degree is not always desirable. For instance it is known that homogeneous networks, consisting of nodes with a similar degree, are more easy to synchronize [16]. In the light of the self-organizing powers of adaptive networks it is therefore reasonable to ask whether one can construct a network that self-organizes towards a synchronizable topology. This question is addressed in a paper by Zhou and Kurths [80], who study an adaptive network of coupled chaotic oscillators. They consider a system of N coupled oscillators with

dynamic state $x_i(t)$

$$\dot{x}_i = F(x_i) + \sum_{j=1}^N G_{ij} [H(x_j) - H(x_i)], \quad (1.8)$$

where $F(x)$ describes the local oscillatory dynamics of each node and $H(x)$ is a linear output function. The coupling matrix G_{ij} can be separated into the stationary adjacency matrix A_{ij} and a time dependent node-specific connection strength V_i , i.e. $G_{ij}(t) = A_{ij}V_i(t)$. While the links in the network remain fixed, the connection strength V_i experienced by a node i increases according to

$$\frac{d}{dt} V_i = \gamma \frac{\Delta_i}{1 + \Delta_i}, \quad (1.9)$$

where $\gamma > 0$ is an adaptation parameter and $\Delta_i = |H(x_i) - (1/k_i) \sum_j A_{ij} H(x_j)|$ measures the difference between the state of oscillator i and the state of its neighbors.

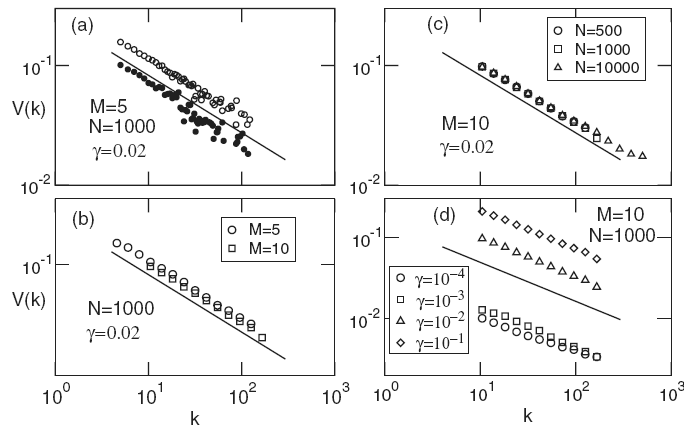
Note, that this rule implies that the connection strength between nodes in different states are strengthened. This is exactly the opposite of the adaptation rule proposed by Ito and Kaneko. Consequently, the adaptive self-organization drives the network into the direction of a more homogeneous topology, and thereby enhancing synchronizability. By numerical simulation of two example systems (Rössler oscillators and an ecologically inspired model [6]) Zhou and Kurths show that the simple adaptive rule enables synchronization in networks that, by several orders of magnitude, exceed the size of the largest comparable random graph that is still synchronizable.

Another hallmark of adaptive networks that reappears in the work of Zhou and Kurths is the emergence of power laws. They show that in the synchronized state the incoming connection weights V_i scale with the degree k_i of the corresponding node as

$$V(k) \sim k^{-\theta}. \quad (1.10)$$

The exponent $\theta = -0.48$ of this power law appears to be universal, i.e., it is independent of the parameters in the model (s. Fig. 1.3). The authors point out that this universal behavior arises because of a hierarchical transition to synchronization. In this transition the nodes of the highest degree are synchronized first. Nodes of lower degree are synchronized later and therefore experience the increase in coupling strength for a longer time.

Let us remark that the results reported in this section indicate that there could be a subtle connection to the mechanism described by Bornholdt and Rohlf [9]. The results of Ito and Kaneko show that there is a scale separation between the dynamics of the network (involving states and topology)



Tab. 1.3 The adaptive network of coupled oscillators studied by Zhou and Kurths organizes towards a topology in which the incoming weight V_i is a power law of the nodes degree k . The exponent $\theta = -0.48$ is independent of (a) the specific type of oscillator under consideration, (b) the mean degree M , (c) the size of the network and (d) the adaptation parameter γ . Source: Zhou and Kurths, *Phys. Rev. Lett.* **96**, 164102, 2006 [80] Fig. 2.

and the timescale on which the emergent properties of the nodes change. In other words the turnover time for a node of high degree to become a node of low degree is many orders of magnitude larger than the time required for the rewiring of individual links. In contrast to other models this time scale separation is not evident in the rules of the system but emerges from the dynamics. One can suspect that this time scale separation could arise because of the presence of a phase transition at which the turnover time diverges. In the light of the findings described in the previous section it is conceivable that an adaptive network could self-organize towards such a phase transition. However, more investigation in this direction are certainly necessary to verify whether this is indeed the case.

1.6

Cooperation in games on adaptive networks

The term ‘division of labour’ used in the previous section already suggests a socio-economic reading. Indeed, socio-economic models are perhaps the most fascinating application of adaptive networks so far. In this context the nodes represent agents (individuals, companies, nations, ...) while the links represent social contacts or, say, business relations. In contrast to other systems considered so far agents are in general capable of introspection and planning.

For this reason the exploration of socio-economic systems is invariably linked to game theory.

1.6.1

Elevated levels of cooperation

One of the central questions in game theory is how cooperation arises in populations despite the fact that cooperative behaviour is often costly to the individual. A paradigmatic game which describes advantageous but costly cooperation is the prisoner's dilemma. In this game two players simultaneously chose between cooperation and defection. From the perspective of a single player choosing to defect always yields a higher payoff regardless of the action of the opponent. However, the collective payoff received by both players is the lowest if both players defect and the highest if both cooperate.

In models, the action a player takes is determined by its strategy, which comprises of a lookup table, that maps the information from a given number of previous steps to an action, complemented by additional rules for the initial rounds where no such information is available. In order to study the evolution of strategies one considers a group of players with initially heterogeneous strategies. From this group pairs of players are picked which participate in the game. Depending on the model the strategies can change according to a set of evolutionary rules, such as replication and mutation of successful players, or by behavioral adaptation, such as optimization of the strategy by trial and error or learning from successful neighbors.

While the pool of players is often considered to be well-mixed, spatial games in which the players are arranged on a static network with links that represent possible games have been studied for some time (e.g. [55]). More recently games on adaptive networks have come into focus. In these games the players are able to change their topological position, for example by cutting links to defectors. The prisoner dilemma game on adaptive networks has been studied in [19,20,56,81,82]. An adaptive version of the closely related Snowdrift game was investigated in [63] and a more realistic socio-economic model involving taxes and subsidies was discussed in [47]. In the results presented in these papers the two common themes discussed above, namely the robust topological self-organization and the associated appearance of power laws, reappear and have been noted by many authors. For instance the formation of realistic networks topologies characterized by scale-free degree distributions, small diameter and high clustering coefficient is discussed in detail by Ren et al. [63] and Eguíluz et al. [20].

From an applied perspective it is interesting that elevated levels of cooperation are reported in all papers cited above. The mechanism that promotes cooperation in adaptive networks becomes apparent when one considers the

interaction between the players and their neighbourhood. In all games on networks the local neighbourhood acts as an infrastructure or substrate from which payoffs are extracted. The quality of this infrastructure depends on topological properties such as the degree or the number of cooperators in the neighbourhood. In an adaptive network a player can shape this neighbourhood by its own actions. Thereby the neighbourhood becomes an important resource. The rules of the games are generally such that selfish behaviour degrades the quality of this resource as neighbours are able to respond by adaptive topological changes, say, cutting or rewiring a link to a defector. This feedback may be regarded as a 'topological punishment' of the defecting player which decreases its ability to generate payoff. It is well known that the option to punish defectors generally increases the level of cooperation. Nevertheless, the possibility of 'topological punishment' emphasizes an important point: Additional degrees of freedom – and therefore additional complexity – can arise from the topology. This effectively increase the local complexity in games. In an adaptive network an agent not only has to optimize its payoff in order to be successful, but also has to maintain, and possibly improve, the quality of its topological environment. Thus, the 'quality' of the local topology is an important asset for the agents [55]. It can therefore be suspected that the elevated levels of cooperation that are observed in games on adaptive networks are achieved by means of topological rewards/punishments between the players. The adaptive nature of social networks in populations may therefore play an important role in the maintenance of cooperative behavior.

A rigorous investigation in the mechanism that promotes cooperation on adaptive networks is presented by Pacheco et al. [56]. In the limit in which the topological dynamics is much faster than the evolution of strategies the authors show that the prisoner dilemma on an adaptive network can be mapped to a game in a well mixed population. However, this 'renormalized' game is not a prisoner dilemma; the mapping effectively changes the rules of the game so that the prisoner dilemma is transformed into a coordination game. This explains the elevated levels of cooperation since the cooperative behaviour is naturally favoured in the coordination game.

In addition to elevated levels of cooperation, games on adaptive networks exhibit some other effects which we have already encountered in the previous section. For example, an effect that is reminiscent of the spontaneous 'division of labour' and the emergence of social hierarchies was observed in [20, 81, 82]. Another observation is that the approach to the final state is marked by large avalanches of strategy changes which exhibit power-law scaling [19, 20, 81]. Such scaling behaviour again is an indicator of self-organized critical behaviour.

Based on the results presented in the previous sections, it is reasonable to assume that these effects arise due to adaptive topological self-organization

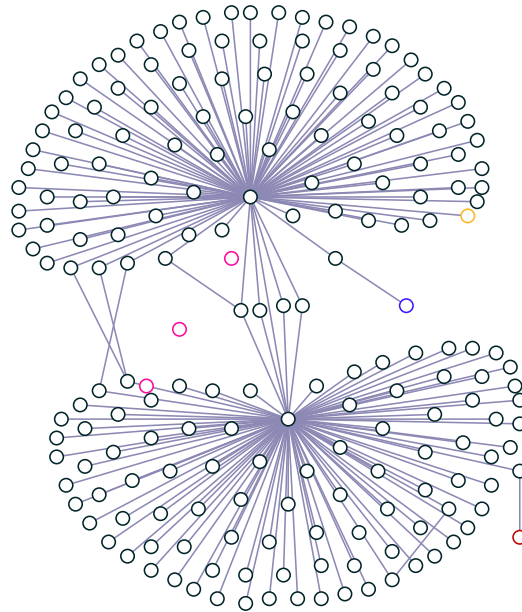
sensu Ito and Kaneko [35]. However, this conclusion is not inevitable. First of all, in these papers the adaptive interplay between the network state and topology stops at some point as the network freezes in a final stationary configuration, in which the players cannot increase their payoff further by altering their strategy or their local topology – a so-called ‘network Nash equilibrium’. It is therefore not clear whether the different social classes observed in the simulations arise from the same mechanism as described in the model of Ito and Kaneko. As another possible explanation the network could have reached an absorbing state, freezing the network and thus fixing local topological heterogeneities in some otherwise transient state. Furthermore, in contrast to the model of Ito and Kaneko the individuals in the games discussed here are not initially identical. Moreover, the strategies as well as the local topology changes discontinuously in a stochastic fashion. It is therefore not entirely clear that the observed hierarchies arise truly spontaneously and not from inhomogeneous initial conditions or large random fluctuations. In order to establish that division of labor and hierarchies can indeed arise spontaneously in adaptive social networks further studies are certainly necessary.

It is interesting to note that the adaptive nature of a network is not always apparent on the first glance. For instance Paczuski et al. [58] study the minority game on a fixed network. In this non-cooperative game each agent makes a decision between two alternatives. The agents who decide for the alternative chosen by the minority of agents are rewarded. The decision of the agent depends on its own decision in the previous round as well as on the decision of its immediate neighbours in the network during that round. As in the prisoner dilemma the strategy of an individual agent can be described by a lookup table that is allowed to evolve in time to maximize success. Despite the fact that the game is seemingly played on a static network Paczuski et al. observe all the hallmarks of adaptive networks described above. This enigma is resolved by noting that the evolution of the strategies in the lookup tables can effectively change the nature of the links in the network. In particular the lookup tables can evolve to such a state that the decision of certain neighbours in the network is ignored entirely [57]. This means that even though the network itself is static the effective degree, which is experienced by the nodes, can change over time. Therefore the network is after all adaptive.

1.6.2

Struggle for topological position

While adaptive networks can add realism to previously studied games like the prisoner’s dilemma, they also give rise to an entirely new class of games. In these games the players do not try to maximize an abstract payoff, but instead struggle to achieve an advantageous topological position on the network. For



Tab. 1.4 In the paper of Holme and Ghoshal, agents compete for a position of high centrality and low degree. This figure shows that complex global topologies are formed. In the figure three classes of nodes can be identified. Most nodes suffer from a low centrality, while others gain high centrality at the cost of having to maintain a large number of links. Only a small class of ‘VIP’ nodes manage to achieve both high centrality and low degree. Source: Holme and Ghoshal, *Phys. Rev. Lett.* **96**, 098701, 2006 [32] Fig. 2b.

example, in a social network a position of high centrality is certainly desirable. The struggle for such a position is studied in models by Rosvall and Sneppen [65–67]. This set of models describes the formation of a communication network between social agents. As an interesting feature the communication provides the agents with metainformation about the network structure. In this way information about the topology is transmitted through the same network the agents try to optimize. Rosvall and Sneppen show that the proposed model leads to the self-assembly of an effective communication network. While the authors are primarily concerned with the impact of individual behavior, such as lying about the actuality information, on the network, we can at least suspect that the self-organization of the network is governed by the same adaptive interplay between state and topology that we have encountered in the previous sections.

In a related model by Holme and Ghoshal [32] the agents attempt to achieve a position of high centrality while minimizing the number of contacts they have to maintain. In every step an agent has the option to disconnect one of

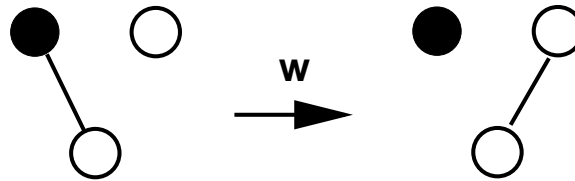
its links. Independently it has the option to form one new link to an agent that is up to two steps away. The exact target to which a link is established or cut depends on the agents connection or disconnection strategy, respectively. Both of these strategies are chosen by ranking a fixed set of criteria, which correspond to choosing the node with the minimal or maximal degree, minimal or maximal centrality, a random node or no node at all. After a certain time, the strategy of a random node is updated by copying the highest-ranked criterion from the most successful agent.

Holme and Ghoshal show in simulations that the system exhibits long periods of stability where one strategy is dominant. These periods are interrupted by sudden invasions of a different strategy. Apparently, no steady state is approached so that the successional replacement of the dominant strategy continues in the long term behavior. An interesting feature of the model is that it transiently gives rise to highly nontrivial topologies. Figure 1.4 shows an example of such a topology. The shown topology is complex in the sense that it is immediately evident that it is not random or regular, but possesses a distinct structure. Note, that three distinct classes of nodes can be recognized in the figure. In particular there is a class of agents who achieve the goal of being in a position of high centrality and low degree. However, while a spontaneous division of labour is evident, there is no de-mixing of classes: A node holding a position of low degree and high centrality at a certain time does not have an increased probability of holding such a position at a later time. Note also that the node's centrality that enters into the model is a global property. Therefore the emerging topologies are not organized based on local information alone.

1.7

Dynamics and phase transitions in opinion formation and epidemics

Above we have mainly been concerned with systems in which the state of the network changes much faster or much slower than the evolution of the topology. In systems that exhibit such a time scale separation only the averaged state of the fast variables can affect the dynamics of the slow variables and therefore, the dynamical interplay between the time scales will, in general, be relatively weak. In contrast, new possibilities open up in systems in which the evolution of the topology takes place on the same timescale as the dynamics on the network. As dynamical variables and topological degrees of freedom are directly interacting a strong dynamical interplay between the state and topology becomes possible. One might say that information on the dynamics of the state can be stored in and read from the topology and vice versa. In the study of this interplay we can no longer make use of the time scale separation. Nevertheless it is still possible to analyze and understand



Tab. 1.5 Adaptive rewiring in an epidemic network. Nodes can be either susceptible (open circle) or infected (filled circle). With a certain probability w per unit time, susceptibles break their link to the infected individuals and form a new connection to another randomly selected susceptible.

the dynamics on the network by using the tools of nonlinear dynamics and statistical physics. Depending on the language of description the qualitative transitions in the dynamics and topology then become apparent in the form of either bifurcations or phase transitions.

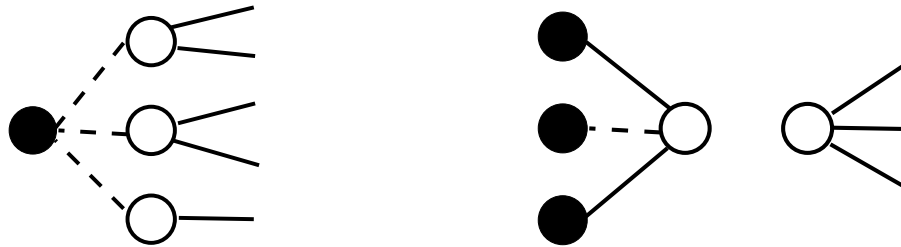
In the following we discuss some insights on such phase transitions that have been obtained in the investigation of contact processes, such as opinion formation and epidemic spreading. Let us emphasize that these new classes of phase transitions differ from the ones usually observed in physics, since they involve topological degrees of freedom of the system.

1.7.1

Epidemiological models

A simple framework in which the dynamical interplay can be studied is offered by contact processes, which describe the transmission of some property, such as information, political opinion, religious belief or epidemic infection along the network connections. One of the most simple models in this class is the epidemiological SIS model. This model describes a population of N individuals forming a social network with K bidirectional links. Each individual is either susceptible (S) to the disease under consideration or infected (I). A susceptible individual in contact with an infected individual becomes infected with a fixed probability p per unit time. Infected individuals recover at a rate r immediately becoming susceptible again. If considered on a static network the SIS model has at most one dynamical transition. Below the transition only the disease-free state is stable, while above the transition the disease can invade the network and approaches an endemic state.

The spatial SIS model can be turned into an adaptive network if an additional process is taken into account: susceptible individuals are allowed to protect themselves by rewiring their links [29]. This takes into consideration that humans tend to respond to the emergence of an epidemic by avoiding contacts with infected individuals. By changing their local contact structure

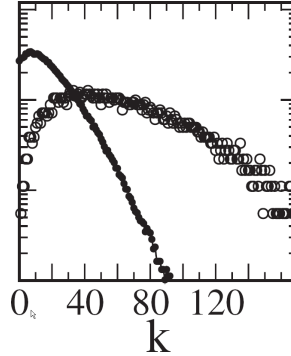


Tab. 1.6 Two main effects of adaptive rewiring. Similar to Fig.1.5 nodes can be either susceptible (open circles) or infected (filled circles). Edges that will be reconnected in the next time step are indicated as dashed lines. Left: (isolation) due to rewiring infected nodes become isolated, which results in an effective reduction of transmission. Right: (mixing) the susceptible to the right is not connected to the herd of infection and therefore without rewiring would be safe. Rewiring, however, can connect previously non-connected or isolated compartments of the network, and so allows for a transmission of the disease into the previously safe component.

individuals can cause changes in the topology of the network as a whole. Such structural changes can have a strong effect on the dynamics of the disease, which in turn influences the rewiring process; finally resulting in the typical complicated mutual interaction between a time varying network topology and the dynamics of the nodes.

Such a scenario was studied by Gross et al. [29]. In their model with probability w a given susceptible breaks the link to an infected neighbor and forms a new link to another randomly chosen susceptible. Double- and self-connections are not allowed to form in this way. As was shown in Gross et al. [29] this simple intuitive rewiring rule for the network connections has a profound impact on the emerging network. Even for moderate rewiring probabilities it is able to change the dynamics of the system qualitatively and generates specific network properties such as a wide degree distribution, assortative degree correlations and the formation of two loosely connected sub-compartments. The dynamical consequences are the emergence of new epidemic thresholds (corresponding to first order transitions), the coexistence of multiple stable equilibria (leading to hysteresis), and the appearance of an oscillatory regime, all of which are absent on static SIS networks.

A first measure for the effect of adaptive rewiring is given by the threshold infection probability p^* that is necessary to maintain a stable epidemic. On a random graph without rewiring ($w = 0$) the basic reproductive number, which denotes the secondary infections caused by a single infected node on an otherwise susceptible network is $R_0 = p\langle k\rangle/r$, where $\langle k\rangle = 2K/N$ is the mean degree of the nodes. Demanding that exactly one secondary infection is caused yields $p^* = r/\langle k\rangle$. If rewiring is taken into account a single infected node will on average loose a constant fraction w of its links. Therefore the degree of such a node can be written as $k(t) = \langle k\rangle \exp(-wt)$, where t is the



Tab. 1.7 Spontaneous 'division of labour' in the model of Gross et al. [29]. Plotted is the degree distribution ρ_k for susceptibles (circles) and infected (dots). Two topologically distinct populations of nodes emerge and are characterized by low and high degree k respectively. Parameters: $N = 10^5$, $K = 10^6$, $w = 0.3$, $r = 0.002$, $p = 0.008$.

time since infection. By averaging over the typical lifetime $1/r$ of an infected node, we obtain the effective number of links $\langle k \rangle (1 - \exp(-w/r))r/w$ and therefore the threshold infection rate

$$p^* = \frac{w}{\langle k \rangle (1 - \exp(-w/r))}. \quad (1.11)$$

Note, that this corresponds to $p^* = r/\langle k \rangle$ for $w = 0$, but $p^* = w/\langle k \rangle$ for $w \gg r$. In this sense a high rewiring rate can act as a very efficient protection and can significantly increase the epidemic threshold and thereby reduce the prevalence of the epidemics (see also Fig.1.6 left). In comparison, the effect of adaptive rewiring on the topology is more subtle. Even if a component of the network manages to disconnect itself from all infected, it will generally not stay disease free since rewiring introduces an ongoing mixing in the network that can reestablish bridges to the disconnected component (see Fig.1.6 right). In this sense in the adaptive social network there are no "safe havens".

However there are further topological effects. Consider first the trivial case in which rewiring is independent of the state of the nodes. In this case the degree distribution becomes Poissonian and the average degree k_{nn} of the next neighbors of a given node is independent of the degree k , as one would expect in a static random graph. Now, assume that the adaptive rewiring rule described above is used, but the local dynamics is switched off, $r = p = 0$. In this case the density of infected, i , and susceptibles, $s = 1 - i$, stays constant. However, the number of SI-links is reduced systematically over time until the network has split into two disconnected clusters, one of which is occupied by infected while the other is occupied by susceptibles. Assuming that we start with a random graph, the per-capita densities of SS-, II- and SI-links

are initially $l_{SS} = s^2 \langle k \rangle / 2$, $l_{II} = i^2 \langle k \rangle / 2$ and $l_{SI} = \langle k \rangle / 2 - l_{SS} - l_{II} = si \langle k \rangle$, respectively. With adaptive rewiring, in the stationary state all SI links have been converted into SS links so that $l_{SS} = (1 - i^2) \langle k \rangle / 2$ and $l_{SI} = 0$. Consequently, susceptibles and infected assume different degree distributions ρ_k , in which the mean degree of a susceptible node is $\langle k_S \rangle = (1 + i) \langle k \rangle$ and the mean degree of an infected node is $\langle k_I \rangle = i \langle k \rangle$. While both clusters are still individually Poissonian, the susceptible cluster has a higher connectivity. Since k_{nn} is independent of k in each of the two clusters, the degree correlation within each cluster vanishes. However a considerable net degree correlation $r_{\text{corr}} > 0$ (see section 1.2.1) can arise if both clusters are considered together because k_{nn} is larger for the susceptible cluster.

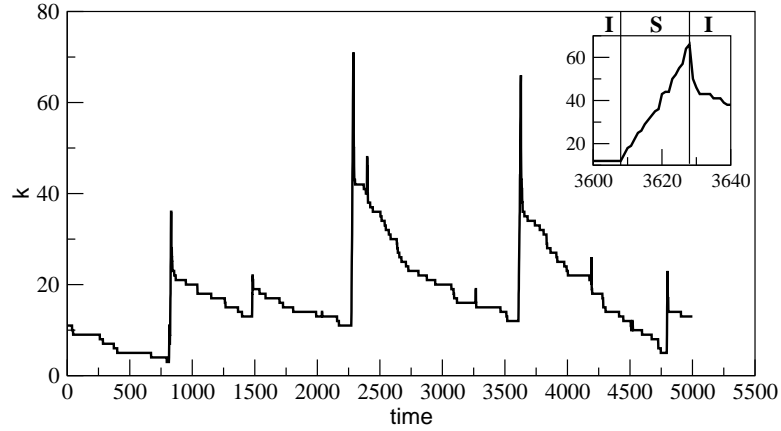
Finally, consider the case with both adaptive rewiring and epidemic dynamics (Fig. 1.7). Even though rewiring is not fast enough to separate infected and susceptibles completely, it still structures the system into two loosely connected clusters of susceptibles and infected (e.g., $l_{SI} \approx 0.01 \langle k \rangle$ in the figure). While inter-cluster connections are continuously removed by rewiring, new ones are formed by recoveries in the infected cluster and infections in the susceptible cluster. With increasing rewiring rate w the degree correlation grows rapidly. Moreover, the mean degree of the susceptibles increases while the degree of the infected decreases slightly. Even more pronounced is the increase in the variance of the degree distribution of susceptibles [29]. This indicates the formation of strongly connected hubs and temporarily isolated nodes, which are rapidly reconnected due to rewiring.

Thus, adaptive rewiring has different antagonistic effects on the spreading of the disease. Locally, rewiring promotes the isolation of infected individuals, which can significantly increase the epidemic threshold. However, in doing so rewiring introduces a mixing of connections in the population so that every herd of infection has the potential to jump into previously unconnected compartments of the network. Finally, over a longer timescale rewiring leads to a build up of links in the susceptible population. In this way a highly connected cluster of susceptibles is formed in which the epidemic rapidly propagates once it manages to invade. Therefore the local effect of rewiring tends to suppress the epidemic while the topological effect promotes it.

The adaptive rewiring of the topology leads to large temporal fluctuations in the degree $k_i(t)$ of a node (see Fig. 1.8). The picture reveals three characteristic phases:

i: *jump upwards*

As long as an individual is susceptible, it rapidly obtains new links due to the rewiring activity of the other susceptibles. This results in a fast increase in the degree of the node which is approximately linear in time, $\dot{k}(t) = w l_{SI}$.



Tab. 1.8 Dynamics of a single node in the fixed point regime. Plotted is the local degree $k(t)$ of a single node as a function of time. Obviously, even in the steady state there are locally strong fluctuations in the topology. The picture reveals three characteristic phases in the dynamics of the degree of a single node (see text). Inset shows an enlarged view of the fast temporal spikes when the node under consideration has recovered and is in the state S. Parameters $N = 10^5$, $L = 10^6$, $w = 0.4$, $p = 0.008$, $r = 0.002$.

ii: *jump downwards*

Once a susceptible has become infected, very rapidly all susceptible neighbors rewire and cut the connection. This results in an even faster reduction in the degree directly after infection, until the infected has only infected neighbors.

iii: *decay*

Eventually the infected neighbors of the node under consideration recover and immediately rewire to new susceptible neighbors. This results in the third phase, characterized by a slow exponential decay in the degree, $\dot{k}(t) \sim -wk$. This phase continues until the infected node itself recovers and phase (i) is reinitialized.

Note that the fast phases (i) and (ii) give rise to the characteristic spikes in the temporal dynamics $k_i(t)$ of the degree of a single node, while the slow phase (iii) accounts for the long exponentially decaying segments (see Fig.1.8). In this way, by ongoing changes in the local degree of each node, a complicated dynamical equilibrium can form in which the average number of inter- and intra-cluster links as well as the density of susceptibles and infected stays constant.

In order to capture the dynamics of the adaptive network it is useful to consider a low dimensional model. From the discussion above both the dynamic state and the topological structure of the network can be characterized

in terms of the density of infected i and the second network moments: the density of links between susceptibles l_{SS} and the density of links between susceptibles and infected l_{SI} . To describe the time evolution of these variables Gross et al. [29] and subsequently also Zanette [79] apply the moment closure approximation proposed by [41]. In this pair approximation the density of all triples l_{abc} in the network with the respective states $a, b, c \in \{S, I\}$ are approximated by $l_{abc} = l_{ab}l_{bc}/b$, i.e., as the product of the number of ab -links l_{ab} with the probability l_{bc}/b that a given node of type b has a bc -link. This yields for instance for the density of S-S-I chains

$$l_{SSI} \approx \frac{2l_{SS}l_{SI}}{s}, \quad (1.12)$$

where $s = 1 - i$ is the density of susceptibles.

Using this approximation, straightforward calculation leads to a system of three coupled ordinary differential equations

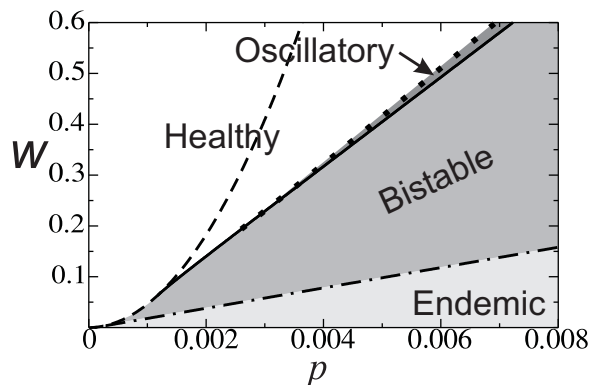
$$\frac{d}{dt}i = pl_{SI} - ri \quad (1.13)$$

$$\frac{d}{dt}l_{II} = pl_{SI} \left(\frac{l_{SI}}{s} + 1 \right) - 2rl_{II} \quad (1.14)$$

$$\frac{d}{dt}l_{SS} = (r + w)l_{SI} - \frac{2pl_{SI}l_{SS}}{s}. \quad (1.15)$$

The first term in Eq. (1.13) describes the infection of susceptible individuals, while the second term describes recovery. These two processes also effect the dynamics of the links. The first term in Eq. (1.14) corresponds to the conversion of SI links into II links as a result of new infections while the second term represents the conversion of II links into SI links as a result of recovery. Equation (1.15) is analogous except for the fact that the conversion of SI links into SS links by rewiring has been taken into account. Note that in Eqs. (1.13 - 1.15) three dynamical variables are necessary, while the system-level dynamics of the standard (non-adaptive) SIS model can be captured by only one variable. This illustrates that in the adaptive model two topological degrees of freedom communicate with the dynamics of the nodes.

Investigation of the low dimensional model reveals a complex bifurcation structure. Without rewiring, there is only a single, continuous dynamical transition, which occurs at the well known epidemic threshold, p^* . As the rewiring is switched on, this threshold increases in perfect agreement with Eq. (1.11). While the epidemic threshold still marks the critical parameter value for the invasion of new diseases another, lower threshold, corresponding to a saddle-node bifurcation, appears. Above this threshold an already established epidemic can persist (endemic state). In contrast to the case without rewiring the two thresholds correspond to discontinuous (1st order) tran-



Tab. 1.9 Two parameter bifurcation of the adaptive epidemiological network studied by Gross et al. Bifurcations divide the parameter space into regions of qualitatively different dynamics, in dependence of the infection probability p and the rewiring rate w . In the white and light grey regions there is only a single attractor, which is a healthy state in the white region and an endemic state in the light grey region. In the medium grey region both of these states are stable. Another smaller region of bistability is shown in dark grey. Here, a stable healthy state coexists with a stable epidemic cycle. The transition lines between these regions correspond to bifurcation. The dash-dotted line marks a transcritical bifurcation that corresponds to the threshold at which the epidemic can invade the disease free system. The region in which an established epidemic can remain in the system is bounded by a saddle-node bifurcation (dashed), a Hopf bifurcation (continuous) and a fold bifurcation of cycles (dotted). The saddle-node and transcritical bifurcation lines emerge from a cusp bifurcation at $p = 0.0001, w = 0$. Source: Gross et al., Phys. Rev. Lett. **96**, 208701, 2006 [29] Fig. 4.

sitions. Between them a region of bistability is located, in which the healthy and endemic state are both stable. Thus, a hysteresis loop is formed.

By numerical simulations Gross et al. show that the presence of a hysteresis loop and first order transitions is a generic feature of the adaptive model and can be observed at all finite rewiring rates (see Fig.1.9). While increasing the rewiring rate hardly reduces the size of the epidemic in the endemic state, the nature of the persistence threshold changes at higher rewiring rates. First, a subcritical Hopf bifurcation, which gives rise to an unstable limit cycle replaces the saddle-node bifurcation. At even higher rewiring rates this Hopf bifurcation becomes supercritical. Since the emerging limit cycle is now stable, the Hopf bifurcation marks a third threshold at which a continuous transition to oscillatory dynamics occurs.

Thus, at high rewiring rates the adaptive SIS model in [29] can approach an oscillatory state in which the prevalence of the epidemic changes periodically. The oscillations are driven by the two antagonistic effects of rewiring mentioned above. On the one hand rewiring isolates the infected and thereby reduces the prevalence of the disease. On the other hand the rewiring leads to an accumulation of links between susceptibles and thereby forms a tightly

connected cluster. At first the isolating effect dominates and the density of infected decreases. However, as the cluster of susceptibles becomes larger and stronger connected a threshold is crossed at which the epidemic can spread through the cluster. This leads to a collapse of the susceptible cluster and an increased prevalence which completes the cycle. While this cycle exists only in a narrow region (Fig. 1.9) in the model described above, the parameter region in which the oscillations occur and the amplitude of the oscillations are enlarged if one takes into account that the rewiring rate can depend on the awareness of the population and therefore on the prevalence of the epidemic [30].

In the adaptive SIS model the hallmarks of adaptive networks discussed above reappear: The isolation of infected and the emergence of a single tightly connected cluster of susceptibles is an example of the appearance of global structure from local rules. Moreover, the mechanism that drives the oscillations is reminiscent of the self-organization to criticality discussed in Sec. 1.4. The rewiring rule that is used in the adaptive SIS model establishes connections between nodes in identical states and severs connections between different states. Stated in this way the rewiring rule reminds of the model of Ito and Kaneko (see Sec. 1.5) in which connections between similar nodes are strengthened and others weakened. This analogy suggests that topologically different classes of nodes could emerge from the dynamics of the network. Indeed, Fig. 1.7 shows that two classes of nodes appear, which are characterized by different degree distributions. In this case we can identify the classes to consist of infected and of susceptible nodes, respectively. However, here again (similar to the study by Holmes and Goshal section 1.6.2) individuals do not remain in one class forever, but constantly jump between the classes of susceptibles and infected. This means that no de-mixing of classes as in the study by Ito and Kaneko (see section 1.5.1) arises. The example of the adaptive SIS model illustrates that topological degrees of freedom can greatly enrich the dynamics of a system. It is remarkable that of the large number of topological degrees of freedom that exist in the system only two seem to interact strongly with the local dynamics.

For the control of real world diseases the mechanism of adaptive rewiring is beneficial since it increases the invasion threshold and also the persistence threshold for epidemics. However, the topological changes that are inevitably induced as a natural response to an emerging disease are a cause for concern. Vaccination policies depend on a precise knowledge about important network properties. However, as we have shown, the network structure can rapidly change in response to the onset of an epidemic. Thus, the topology at the peak of a major epidemic can be very different from that in the disease-free state. In particular, adaptive rewiring can lead to the formation of a highly volatile cluster of susceptibles which enables the persistence of epidemics even be-

low the epidemic threshold. Further, it can rapidly introduce positive degree correlations with a strong detrimental effect on the effectiveness of targeted vaccination. As a consequence of the natural reaction of the network topology, a disease which seems to be a minor problem while it is rare can be very difficult to combat once it has reached an endemic state. Vaccination levels that may seem sufficient in the healthy state may therefore be insufficient to stop epidemics of major diseases.

Similar findings as in the adaptive network model [29] are also reported in an investigation of a series of models by Ehrhardt et al., which describe the spreading of innovation and related phenomena on an adaptive network [21]. In these models knowledge spreads through a network in which links are continually formed and destroyed. In the case in which new links are preferentially formed between nodes with a similar level of knowledge, the system constitutes an adaptive network. As in the network studied by Gross et al. the dynamics of the system can be captured by a low dimensional model, that is in this case derived by means of a mean field approximation. Based on explicit simulations and the investigation of the low dimensional approximation, Ehrhardt et al. show that the model exhibits discontinuous phase transitions, which form a hysteresis loop. The authors argue that evidence for such dynamics can be observed in empirical data on social networks.

1.7.2

Opinion formation

Another approach to the dynamics of adaptive networks is offered by the tools of statistical physics, which can reveal critical points in the form of phase transitions. One example of such a phase transition is presented in a paper by Holme and Newman [33], which focuses on opinion formation in populations. Specifically the paper considers the case of opinions, such as religious belief, for which the number of possible choices is only limited by the size of the population. Disagreeing neighbours manage to convince each other with probability ϕ or rewire their connections with probability $1 - \phi$. This ultimately leads to a consensus state in which the network is decomposed into disconnected components, each of which consist of individuals who hold a uniform opinion. For $\phi = 0$ opinions never change, so that the final distribution of opinions matches the initial distribution. For $\phi = 1$ no connections are rewired, so that the number of opinions in the consensus state can not exceed the number of disconnected components that already existed in the initial network. Applying a finite-size scaling analysis Newman and Holme are able to show that between these extremes a critical parameter value ϕ_c is located, at which a continuous phase transition takes place. At this transition a critical slowing down is observed, so that the network needs a particularly long

time to reach the consensus state. In the consensus state the distribution of followers among the different beliefs approaches a power-law.

The phase transition identified by Holme and Newman probably holds the key to the findings reported in [25]. In this paper Gil and Zanette investigate a closely related model for the competition between two conflicting opinions. Here conflicts are settled by convincing neighbours or cutting links. It is shown that a critical point exists at which only very few links survive in the consensus state. Based on the previous results it can be suspected that this is a direct consequence of the critical slowing down close to the phase transition. In this region the long time that is needed to settle to the consensus state might result in a very small number of surviving links. In this region the long time that is needed to settle to the consensus state implies that also disconnection events take place for a long time and consequently might result in a very small number of surviving links.

1.8

Summary, Synthesis and Outlook

In this paper we have reviewed a selection of recently proposed models for adaptive networks [28]. These examples illustrate that adaptive networks arise in a large number of different areas including ecological and epidemiological systems; genetic, neuronal, immune networks; distribution and communication nets and social models. The functioning of adaptive networks is currently studied from very different perspectives including nonlinear dynamics, statistical physics, game theory and computer science.

1.8.1

The four hallmarks of adaptive networks

Despite the diverse range of applications from which adaptive networks emerge, we have shown that there are a number of hallmarks of adaptive behaviour that recurrently appear:

- **Robust topological self-organization**
The adaptive feedback provides a very robust mechanism for global self organization based on local rules. It enables the agents that form the network to robustly organize into a state with special topological or dynamical properties. In particular we have discussed the example of self-organization to dynamical critical states, which frequently goes together with the appearance of power-law distributions. This form of self organization is probably important for networks that need to remain in or close to the critical change while they undergo topological changes,

such as neural or genetic networks. However, it can possibly also have an impact on many other systems, e.g., ecosystems.

- **Spontaneous emergence of hierarchies and division of labour**
The self-organizing properties of adaptive networks may explain many of the peculiar topological properties that we observe in the real world. One of these properties is the existence of different classes of nodes. In adaptive networks classes of topologically and functionally distinct nodes can arise from an initially homogeneous population. In certain models a 'de-mixing' of these classes is observed, so that nodes that are in a given class generally remain in this class. Such a spontaneous 'division of labor' that has been observed in adaptive networks can in principle hold the key to understanding the emergence of ruling in early societies. Moreover, the related spontaneous formation of hierarchies can have important dynamical effects, such as enhancing the synchronizability of the network.
- **Additional local degrees of freedom**
A genuine adaptive network effect which is not related to global topological organization is the increased number of local degrees of freedom. Agents which form an adaptive network, can affect their local topological environment. Properties of the local topology can therefore be regarded as additional local degrees of freedom.
- **Complex system-level dynamics**
Since information can be stored and read from the topology, the dynamics of adaptive networks involves local as well as topological degrees of freedom. Therefore the dynamics of adaptive networks can be more complex than that of similar non-adaptive models. In particular adaptive networks can give rise to new continuous and discontinuous phase transitions. Furthermore, even very basic models of adaptive networks that are based on simple local rules can give rise to complex global topologies.

In the context of applications, the hallmarks described above can be used as a working guideline: If one of these phenomena is observed in nature one should consider the possibility that it is caused by a, possibly so-far unobserved or not recognized, adaptive network. As was demonstrated in the example of Paczuski et al. [58] the adaptive nature of a network may not always be obvious, but it can be revealed by a direct search. The reverse approach can also be rewarding: In systems which are known to contain an adaptive network it is promising to search for the hallmarks described above.

1.8.2

Adaptive networks: future impacts

Given the evidence that is summarized in this review, we believe that adaptive networks could hold the key for addressing several current questions in many areas of research, but in particular in biology. Adaptive self-organization could explain how neural and genetic networks manage to remain in a dynamically critical state. Spontaneous division of labor could be important for many social phenomena, such as leadership in simple societies, but also for developmental problems such as cell differentiation in multicellular organisms. The capability of adaptive networks to form complex topologies has not been studied in much detail, but it seems to offer a highly elegant way to build up large-scale structures from simple building blocks. A biological example where this certainly plays the role is for instance the growth of vascular networks.

Many important processes have so far mainly been studied only on static networks. However, by doing so important aspects of such systems may be overseen or neglected. Take for example the spread of infectious diseases (section 1.7.1). Currently huge efforts are made to determine the structure of real world social networks. These are then used as input into complicated prediction models, which help to forecast the spread and dynamics of future epidemics (e.g. influenza). However, the most involved model or the best survey of the actual social network is in vain if it is not considered that people may radically change their behaviour and social contacts during a major epidemic.

We want to stress that answers to the questions outlined above would not only enhance our understanding of real world systems comprising of adaptive networks, but could also be exploited in bio-inspired technical applications that self-assemble or self-organize many subunits towards desired configurations. Such strategies are much sought for because many of these artificial systems will soon be too complicated to be easily designed by hand. Thus adaptive network structures may hold the key to provide novel, much-needed design principles and could well radically change the way in which future electrical circuits, production systems or interacting swarms of robots are operating.

1.8.3

Towards a unifying theory of adaptive networks

From an applied point of view it is desirable to compose an inventory of the types of microscopic dynamics that have been investigated in adaptive networks and their impact on system-level properties. Such an inventory could give researchers a guideline as to what kind of phenomena can be expected in natural systems where similar processes are at work. For instance we have

seen that in processes where activity disconnects (i.e., frozen nodes gain links while active nodes lose links) the outcome in general will be a self-organization towards percolation transition where active nodes scale as a power law (e.g., [9, 64]). In contrast, 'like-and-like' processes, which strengthen connections between similar nodes, quite universally seem to give rise to heterogeneous topologies and global structures, possibly scale free networks and the emergence of topologically distinct classes of nodes (e.g., [8, 35]). If however, connections between nodes in different states are strengthened, the outcome will be homogeneous topologies and power-law distributed link weights (e.g., [80]).

In certain places the observations can be supplemented by mathematical insights. For instance in every scale separated system there has to be a discontinuous transition in the fast dynamics in order to maintain an adaptive interplay in the long term evolution of the system. Otherwise the fast dynamics is simply slaved to the slow dynamics. Nevertheless much more information on the dynamics of adaptive networks is necessary to fill the inventory. This information will most likely come from automated numerical studies of large classes of adaptive networks.

We note that the analysis of an adaptive network is not necessarily more involved than that of its static counterpart. While the nodes in static networks generally have different topological neighbourhoods, by contrast, the neighbourhood of nodes in adaptive networks changes over time. Because of this mixing of local topologies the network as such becomes more amenable to averaging and mean field descriptions. However caution is in order, because naive mean field approximations can fail if a spontaneous division of labour occurs in the system and is not taken into account.

Apart from the investigation of further examples of adaptive networks more fundamental work is certainly necessary. The studies reviewed in this paper can only be considered as a first step towards a general theory of adaptive networks. However, some important principles are already beginning to crystallize. The mechanism that drives the robust self-organization towards criticality is quite well understood: The dynamics on the network makes topological degrees of freedom accessible in every node. It thus spreads information on topological properties across the network. The local topological evolution can then react on this information and thus drive the topology to a topological phase transition at which the dynamics on the network is critical. Above we have conjectured that the observed 'division of labour' could be driven by a similar mechanism, characterized by self-organization towards a phase transition at which the critical slowing down of the turnover times between emergent properties of nodes occurs. Moreover, the appearance of topologically distinct classes of nodes is certainly an important factor for the formation of complex topologies. Another factor is probably the dual mechanism de-

scribed at the end of Sec. 1.4 by which global organization of the topology is possible. Finally, the investigations reported in Sec. 1.7 illustrate how topological degrees of freedom, acting as dynamical variables, can give rise to complex system-level dynamics. Thus, the four hallmarks described above seem after all to be connected. It is therefore not unlikely that all of these peculiar properties of adaptive networks can be explained by a single theory describing the transfer of information between state and topology of the network and the subtle interplay between different timescales.

It is interesting to note that all of the effects described above can already arise in adaptive networks in which a scale separation between dynamics on the network and topological evolution exists. However, in order for the forth effect to take place, a threshold in the slow system has to exist at which the dynamics of the fast system changes discontinuously. Otherwise the fast system is slaved to slow dynamics and no dynamical interplay giving rise to more complex dynamics takes place. For this reason the emergence of new phase transitions is much more likely in truly dynamical systems which do not involve a scale separation.

Furthermore, it already becomes clear that certain approaches for studying or even setting up an adaptive network are beneficial. For example, the use of weighted networks is a convenient choice for the analysis of structural changes in adaptive networks. They can be initialized with uniform weights and states plus minor fluctuations. Effectively that means that all oscillators are initially in almost identical states and are connected to all other oscillators with equal strength. That is, initially the nodes form a homogeneous population. However, over the course of the simulation the weight of a large fraction of links approaches zero, so that a distinct network structure emerges. This structure then can be visualized (and analyzed) by only considering links above a certain weight and neglecting all others.

While the study of adaptive networks is presently only a minor offshoot, the results summarized above lead us to believe that it has the potential to grow into a strong new branch of network research. In particular the prospect of a unifying theory and the widespread applications highlight adaptive network as promising area for future research.

1.8.4

Future challenges

Since adaptive networks appear in many different fields and are already implicitly contained in many models a theory of adaptive networks can be expected to have a significant impact on several areas of active research. Future fundamental research in adaptive networks should focus on supplying and eventually assembling the building blocks for such a theory. While it has been

shown that dynamics on the network can make global order parameters locally accessible, this mechanism has only been demonstrated for a few types of local dynamics. Except for these examples it is not clear which set of local rules reveals what kind of global information. Another open question is how exactly the observed 'division of labour' arises and how exactly non-trivial global topologies emerge from the local interactions. Finally, it is an interesting question which topological properties are affected by a given set of evolution rules, so that they can act about topological degrees of freedom.

At present there is a striking discrepancy between the huge areas of science in which adaptive networks regularly appear in models and the small number of papers which are devoted to a detailed investigation of the dynamics of adaptive networks. This discrepancy shows that adaptive networks are at present only a small offshoot of network science, however it also shows that any insights gained in the investigation of adaptive networks will potentially have an immediate impact on a large variety of different fields.

The works reviewed in this paper outline new dynamical phenomena and provide insights into the underlying mechanisms. Nevertheless, many essential building blocks for a theory of adaptive networks are still missing. In particular it is interesting to ask, how many and which topological degrees of freedom, take part in the dynamical interplay and how these depend on the dynamics on the network. Moreover, one would like to understand the dynamical instabilities that give rise to effects like the spontaneous 'division of labor' and complex emergent-level dynamics.

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