Phase synchronization in noisy oscillators with nonisochronicity

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ABSTRACT

We study the synchronization of two nonidentical oscillators with nonvanishing nonisochronicity under the presence of uncorrelated Gaussian noise. To measure the amount of synchronization we calculate the evolution of the phase difference. Without coupling both oscillators rotate with different natural frequencies. Due to the action of coupling this frequency difference is reduced until finally, at a critical coupling strength, synchronization sets in. Under the presence of uncorrelated noise the observed frequency difference is still a monotonically decreasing function of coupling strength but can never become zero due to noise induced phase slips. Here, we show that this usual picture of the transition to synchronization is strongly modified when the oscillators are nonisochronous. In this case the onset of coupling can have different effects and may enlarge or even invert the natural frequency difference of the uncoupled oscillators. Our results can be explained in terms of a noisy particle in a tilted potential.

Keywords: phase synchronization, anomalous synchronization, noise, nonisochronicity, ecological systems

1. INTRODUCTION

The study of synchronization phenomena in populations of interacting oscillators has been a broad and promising field of research in the last decades and is of considerable importance in a variety of physical and biological applications.[1, 2] In practice it is inevitable that the oscillators are nonidentical and vary in their natural frequencies. Such natural disorder is for example always present in biological oscillators and reflects the natural heterogeneity of any living environment. Synchronization then arises as an interplay between coupling and the frequency differences of the oscillators. Of special interest is the phenomenon of phase synchronization in which coupling can overcome the dispersal of natural frequencies and the oscillators mutually adjust their frequencies to a common locking frequency.[3] Phase synchronization is an ubiquitous phenomenon and arises naturally in many areas of physics and living systems. It appears in pairs of mutually coupled limit cycle systems and in phase coherent chaotic oscillators.[4] But phase synchronization is also dominant in systems of many interacting oscillators and has been demonstrated in one or two dimensional lattices and in large sets of globally coupled limit cycle systems[2, 5–10] and chaotic oscillators.[11–13] Biological examples of such synchronization phenomena include synchronous flashing fireflies,[14] firing of neurons and neural networks,[15, 16] the cardio-respiratory system,[17] and oscillating population numbers.[18, 19]

Usually the introduction of coupling simply leads to synchronization between the oscillators. However coupling may also give rise to a plethora of different effects including oscillation death,[8, 20, 21] desynchronization in a short-wavelength bifurcation[22] or dephasing with bursts of amplitude change.[23, 24] Furthermore, any real system is subjected to noisy fluctuations. This is especially true for ecological and biological systems. Here, we focus on a pair of two nonidentical oscillators under the presence of noise and systematically investigate the effects of weak coupling on the frequency distribution between the oscillators. In the noiseless case we report on an unusual transition to synchronization where coupling can enlarge the natural disorder of frequencies and desynchronize the whole ensemble of oscillators.[26]

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2. PHASE SYNCHRONIZATION OF TWO COUPLED OSCILLATORS

In this communication we study the synchronization properties in a pair of two coupled nonidentical oscillators under the presence of noise

$$\dot{\mathbf{x}}_\mathbf{i} = \mathbf{F}(\mathbf{x}_\mathbf{i}; \chi_i) + \epsilon_i(\mathbf{x}_\mathbf{j} - \mathbf{x}_\mathbf{i}) + \eta_i(t), \quad i, j = 1 \ldots 2. \tag{1}$$

To be more specific the systems under investigation obey the following properties:

- In the absence of coupling each oscillator follows its own local dynamics $\dot{\mathbf{x}} = F(\mathbf{x}, \chi)$ where $\mathbf{x} \in \mathbb{R}^n$. Both oscillators have the same functional form but depend on a set of control parameters $\chi = (a, b..)$. We always assume that each oscillator is parameterized either on a limit cycle or on a regime with phase coherent chaos. Thus every, possibly chaotic, oscillator is characterized by a well defined natural frequency which is given by the long term average of phase velocity, $\omega = \overline{\dot{\theta}(t)}$.[3]
- Both oscillators are nonidentical, which is achieved by assigning to each oscillator i an independent value for every control parameter out of the set χ_i, taken from a uniform distribution. In general, the control parameters affect the natural or unperturbed frequency of each oscillator, $\omega_i = \omega(\chi_i)$. Therefore, the natural disorder in control parameters leads to a mismatch of natural frequencies between the oscillators which we also refer to as frequency disorder.
- Both oscillators are diffusively coupled with strength ϵ_i. $C = \mathrm{diag}(c_1, c_2)$ is a diagonal matrix which indicates the interaction in each component of the state vector $\mathbf{x}$. Note, that the coupling strength, ϵ_i, depends on the oscillator i. Thus we allow for nonsymmetric (e.g. unidirectional) coupling. We also assume that even with the onset of coupling each oscillator is still rotating uniformly. This means especially that we don't allow for situations with oscillation death.[8, 9, 20, 21] In practice, this can always be realized if the coupling is restricted to be small enough.
- Each oscillator is under the influence of an additive white noise, $\eta(t)$ taken from a Gaussian distribution with zero mean and standard deviation σ, i.e. $\langle \eta(t)\eta(s)\rangle = \delta(t - s)$. We assume further that the noises of both oscillators are uncorrelated.

Note, that these equations are easily generalized to an ensemble of N coupled oscillators. Synchronization arises as an interplay between the interaction and the frequency mismatch of the oscillators. Thereby, in general, the frequency of each oscillator will be detuned

$$\Omega_i = \Omega_i(\epsilon). \tag{2}$$

We denote the observed oscillator frequency in the presence of coupling with a capital $\Omega(\epsilon)$ in contrast to the natural frequency ω of the uncoupled oscillator, e.g. $\omega = \Omega(0)$. Phase synchronization refers to the fact that with sufficient coupling strength $\epsilon > \epsilon_c$ the two oscillators rotate with the same frequency, $\Omega_i = \tilde{\Omega}$. This definition is used as our main criterion to detect phase synchronization throughout in this communication.

Beside the original system (1) we sometimes use an alternative framework to describe the system dynamics. This is possible since we have assumed that each oscillator is rotating uniformly and therefore it's (uncoupled) dynamics can be described in terms of phase variables $\dot{\theta}_i = \omega_i$. In the case of weakly coupled, nearly identical oscillators, the long-term dynamics of system (1) is given by phase equations of the following form[2, 5, 21, 25]

$$\dot{\theta}_i = \omega_i + \epsilon_i\, \Gamma_{ij}(\theta_j - \theta_i). \tag{3}$$

In this equation, the interaction function Γ_{ij} represents the effects of coupling and, in general, is a 2π-periodic function of the phase difference between the interacting oscillator pairs, $\Delta\theta = \theta_j - \theta_i$. The simplest form of the coupling function arises as the first term in a Fourier expansion of $\Gamma_{ij}(\Delta\theta)$ (Kuramoto model)[2]

$$\Gamma(\theta_j - \theta_i) = \sin(\theta_j - \theta_i). \tag{4}$$

The interaction function can be calculated from the original system in form of the integral[2]

$$\Gamma_{ij}(\Delta\theta) = \frac{1}{2\pi}\int_{\phi=0}^{2\pi} Z_i(\phi) p_{ij}(\phi, \Delta\theta) d\phi. \tag{5}$$

Here, the sensitivity vector $Z_i(\phi)$ describes the phase shift that is induced in oscillator i after a perturbation at phase ϕ, and $p_{ij}(\phi, \delta\theta)$ describes the perturbation of the state of oscillator i with phase ϕ due to the interaction with another oscillator j of phase $\phi + \Delta\theta$.

The evolution of the phase difference $\phi \equiv \theta_2 - \theta_1$ is then determined by the single equation

$$\dot{\phi} = \Delta\omega + [\epsilon_2\, \Gamma_{21}(-\phi) - \epsilon_1\, \Gamma_{12}(\phi)]. \tag{6}$$

It is possible to desribe the evolution of ϕ as the motion of a particle in a 2π-periodic potential $V(\phi)$

$$\dot{\phi} = -\frac{d}{d\phi} V(\phi). \tag{7}$$

with

$$V(\phi) = -\Delta\omega\phi - \int d\phi\, [\epsilon_2\, \Gamma_{21}(-\phi) - \epsilon_1 \Gamma_{12}(\phi)] \tag{8}$$

up to an arbitrary integration constant. As will be shown below, this description via a potential has advantages when additive noise is included. Then the effect of noise is to induce phase slips, or phase jumps, between neighboring minima in the potential.

3. THE NONISOCHRONOUS CASE

We now investigate the mutual entrainment of two deterministic nonidentical phase oscillators which are coupled according to (4) with constant coupling strength

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + \epsilon \sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= \omega_2 + \epsilon \sin(\theta_1 - \theta_2). \end{aligned} \tag{9}$$

The transition to synchronization is depicted in Fig. 1. Both oscillators start out with a natural frequency difference $\Delta\omega = \omega_2 - \omega_1$. With the onset of interaction both oscillator frequencies are detuned (2) and are attracted towards each other. Finally, in the synchronized state, they collide to the single frequency $\tilde{\Omega} = (\omega_1 + \omega_2)/2$. The transition to synchronization can be visualized through a plot of the frequency difference $\Delta\Omega(\epsilon)$ which is a monotonically decreasing function of coupling. When the coupling exceeds a critical value, $\epsilon > \epsilon_c$, the frequency difference disappears $\Delta\Omega(\epsilon) = 0$ and the oscillators are synchronized to a common frequency.

It is also possible to describe the process of synchronization in system (9) analytically. Subtraction of both equations in (9) leads to a single equation for the phase difference

$$\Delta\dot{\theta} = \Delta\omega - 2\epsilon \sin(\Delta\theta) \tag{10}$$

which can simply be integrated to obtain the phase difference as a function of coupling (see for example[27]). This leads to the well known beat frequency of two coupled phase oscillators ($\epsilon < \Delta\omega/2$)

$$\Delta\Omega(\epsilon) = \sqrt{\Delta\omega^2 - 4\epsilon^2}. \tag{11}$$

When the coupling exceeds the synchronization threshold $\epsilon_c = \Delta\omega/2$, then $\Delta\dot{\theta} = 0$ and the phases of both oscillators are related by a fixed phase difference $\sin(\Delta\theta) = \Delta\omega/(2\epsilon)$. Thus, for small coupling levels the state of the system is characterized by the frequency difference $\Delta\Omega(\epsilon)$. With the onset of synchronization the frequency difference disappears and the state of the system can be characterized by the phase lag $\Delta\theta(\epsilon)$.

The process of synchronization in two mutually coupled phase oscillators as described above is particularly simple. It is known for long time that similar phenomena occur if two limit cycle systems are coupled.[3] Interestingly, these ideas can directly be extended to systems with self sustained chaotic dynamics.[4] For these aims

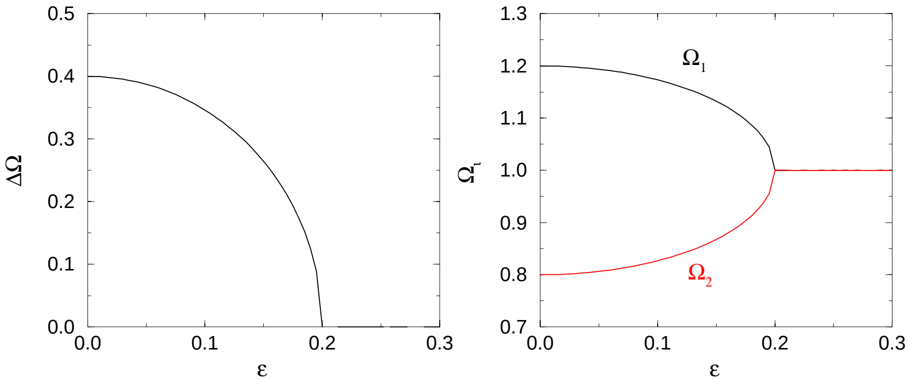

Figure 1. Synchronization of two simple phase oscillators (9). Plotted is the frequency difference $\Delta\Omega(\epsilon)$ (left) and the frequencies of each oscillator $\Omega_{12}(\epsilon)$ (right) as a function of coupling strength, ϵ.

it is necessary to extend the concepts of phase and frequency to the case of a chaotic attractor. This is well established in phase coherent chaotic systems. Take for example the Rössler system[28]

$$\dot{x} = -by - z, \quad \dot{y} = bx + 0.15\,y, \quad \dot{z} = 0.4 + (x - 8.5)z. \tag{12}$$

In the parameter range $b \approx 1$ the motion shows phase coherent dynamics. In this regime a phase can be defined as an angle in (x, y)-phase plane or via the Hilbert-transform.[4] Here, we always estimate the phase of chaotic systems by counting successive maxima, e.g. we locate the times t_n of the n'th local maxima of the y-variable. We define that the phase increases by 2π between two successive maxima and interpolate linearly in between[3]

$$\phi(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \qquad t_n < t < t_{n+1}. \tag{13}$$

Now we explore the transition to synchronization in two mutually coupled Rössler systems (see Fig.2). The oscillators are nonidentical and vary in the value of parameter b. Both oscillators are diffusively coupled in the y variable with equal strength ϵ, e.g. by adding the term $\epsilon(y_{2,1} - y_{1,2})$ in the equation of $\dot{y}_{1,2}$. As can be seen in Fig.2 despite the chaotic amplitudes the transition to the synchronized state is very similar to that of two coupled phase oscillators (9). Due to the interaction both oscillators are detuned and the frequencies approach each other. As a result the frequency difference, $\Delta\Omega(\epsilon)$ decreases monotonically until it becomes zero in the synchronized state.

To summarize, in order to measure the transition to synchronization in a system of two interacting oscillators (1) we identify the frequency of each oscillator in dependence of the coupling strength, $\Omega_i(\epsilon)$. For phase coherent chaotic dynamics this is done by counting the number of local maxima of a chosen variable. We define the frequency disorder as the difference of the observed oscillator frequencies $\Delta\Omega(\epsilon)$. Then phase synchronization is determined by the single criterion that $\Delta\Omega(\epsilon) = 0$.

4. ANOMALOUS SYNCHRONIZATION IN ECOLOGICAL OSCILLATORS

The question arises whether the simple transition to synchronization as exemplified in Figs. 1 and 2 is universal, e.g. whether the frequency disorder $\Delta\Omega(\epsilon)$ is always a monotonically decreasing function of coupling strength. To explore this case we turn to spatially extended ecological systems which are examples for spatio-temporal synchronization in natural systems.[29] Maybe the most intriguing example is Ecology's well known Canadian hare-lynx cycle with hare and lynx populations synchronizing in phase to a collective 10-year cycle over entire Canadia.[19, 30, 32]

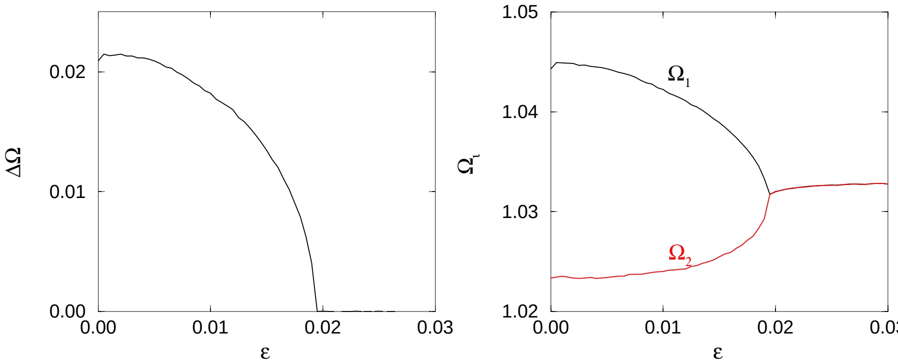

Figure 2. Transition to synchronization in two coupled Rössler systems (12). Plotted is the frequency difference $\Delta\Omega(\epsilon)$ (left) and the individual frequencies of each oscillator, $\Omega_{12}(\epsilon)$ (right) as a function of coupling strength. Parameter values $b_{1,2} = 1.0 \pm 0.01$.

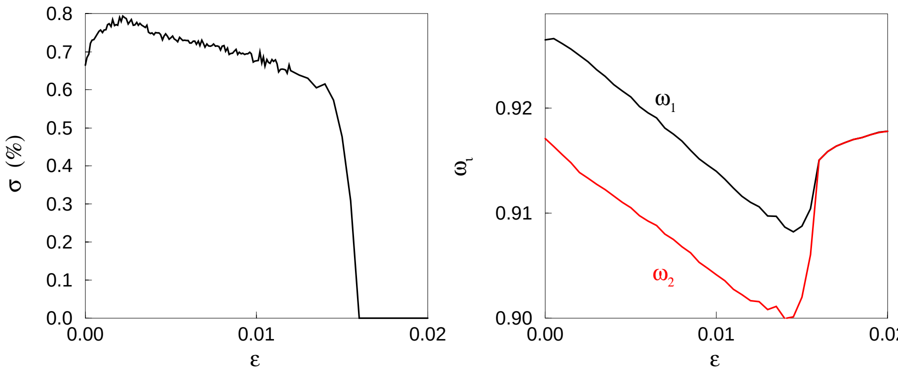

Figure 3. Transition to synchronization in two coupled foodweb models (14). Plotted is the frequency difference $\Delta\Omega(\epsilon)$ (left) and the individual frequencies of each oscillator, $\Omega_{12}(\epsilon)$ (right) as a function of coupling strength. Parameter values ($\alpha_1 = 0.1$, $\alpha_2 = 0.6$, $x_0 = 1.5$, $z_0 = 0.01$, $a = 1$, $c = 10$) and parameter mismatch between both oscillators $b_{1,2} = 0.97 \pm 0.01$.

In order to describe such phenomenon the following model has been proposed[13, 19]

$$\dot{x} = a(x - x_0) - \alpha_1 xy, \quad \dot{y} = -by + \alpha_1 xy - \alpha_2 yz, \quad \dot{z} = -c(z - z_0) + \alpha_2 yz. \tag{14}$$

This model describes a three level "vertical" food chain where the vegetation x is consumed by herbivores y which themselves are preyed upon by the top predator z. In the absence of interspecific interactions the dynamics is linearly expanded around the steady state $(x_0, 0, z_0)$ with coefficients a, b and c that represent the respective nett growth and death rates of each species. Predator-prey interactions are introduced via Lotka-Volterra terms with strength α_1 and α_2. Despite their minimal structure, the equations are able to capture complex dynamics which matches real data for example in the Canadian hare-lynx cycle.[13, 19, 31, 32] In this parameter range the model shows phase coherent chaotic dynamics, where the trajectory rotates with nearly constant frequency in the (x, y)-plane but with chaotic dynamics that appear as irregular spikes in the top predator z. This behaviour of the foodweb model is reminiscent to the Rössler system (12) and therefore one might expect similar synchronization properties in both systems.

To explore this in more detail, in Fig. 3 we calculate the transition to synchronization in two foodweb systems (14) which are coupled in the y-variable exactly as the two Rössler oscillators. The oscillators are nonidentical and vary in the value of their respective consumer death rates b_i. Despite the fact that both, Rössler and foodweb systems, have very similar attractor topology we find fundamental differences in their response to the interaction. Whereas for the two Rössler systems the onset of synchronization is characterized by a monotonically decreasing frequeny difference $\Delta\Omega(\epsilon)$, the two foodweb models show a different behaviour. Here, with increasing coupling the frequency disorder is first amplified leading to a maximal decoherence for intermediate levels of coupling. Only for larger coupling strength frequency disorder is reduced again and synchronization sets in. Thus we observe a counterintuitive effect of coupling which first leads to a desynchronization of the oscillators and to an enlargement of the frequency disorder. We denote this unusual increase of disorder with coupling strength as *anomalous transition to phase synchronization*.[26]

Anomalous phase synchronization is a robust phenomenon and, for example, is also found in large ensembles of oscillators.[26] In general, the strength of anomalous synchronization measured as the maximal gain of frequency disorder can become rather strong extending more than one order of magnitude. As will be shown below, the effect of anomalous synchronization not only allows to control the transition to synchronization but also affects the synchronization threshold. Therefore, it is of potential use for engineering applications and also should play a role in living systems where evolution may have selected parameter sets in such a way as to support biologically advantageous synchronization properties.

5. PHASE DESCRIPTION

In order to analytically describe the transition to synchronization it is useful to turn to a discussion of simple limit cycle systems. Specificly we use two coupled Landau-Stuart oscillators which as a paradigmatic model for synchronization studies provide a universal description of oscillating systems of type (1) when the uncoupled oscillators are not far from the Hopf-bifurcation.[5, 7, 9, 21, 34] Near the onset of oscillations the systems are only weakly nonlinear and the system can be written in polar coordinates as

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = \omega + q(1 - r^2). \tag{15}$$

Here, ω describes the natural frequency of the limit cycle. The term q is the non-isochronicity of the oscillation.[33] It describes the amplitude dependence of the oscillation frequency and therefore is a measure for the shear of phase flow near the limit cycle.

Now consider a system of two diffusively coupled oscillators (15). Assuming that we are dealing with nearly identical oscillators, each oscillator is perturbed in the same way from its limit cycle. In polar coordinates this writes as

$$\begin{aligned} \dot{r_i} &= r_i(1 - r_i^2) + r_i\epsilon\,(\cos\phi - 1) \\ \dot{\theta_i} &= \omega_i + q_i(1 - r_i^2) + \epsilon\,\sin\phi. \end{aligned} \tag{16}$$

Here we have defined the phase difference $\phi \equiv \theta_2 - \theta_1$. Note, that after the radius has relaxed to it's equilibrium values, $\dot{r}_i = 0$ it is possible to write (16) in the generic form of phase equations (3)[2]

$$\dot{\theta}_i = \omega_i + \epsilon\,\left[\sin\phi + q_i(1 - \cos\phi)\right]. \tag{17}$$

Finally we obtain for the phase difference

$$\dot{\phi} = \Delta\omega - \epsilon[2\sin\phi + \Delta q(\cos\phi - 1)], \tag{18}$$

with $\Delta\omega = \omega_2 - \omega_1$ and $\Delta q = q_2 - q_1$. It is straightforward to solve this equation for $\phi(t)$. Here we are interested only in the mean value of angular velocity $\dot{\phi}$ and calculate the time-averaged 'beating period'

$$T = \int_0^{2\pi} \frac{d\phi}{\dot{\phi}}. \tag{19}$$

This can easily be integrated and leads for the mean frequency difference, $\Delta\Omega = \frac{2\pi}{T}$, to

$$\Delta\Omega(\epsilon) = \sqrt{\Delta\omega^2 + 2\epsilon\Delta\omega\Delta q - 4\epsilon^2}. \tag{20}$$

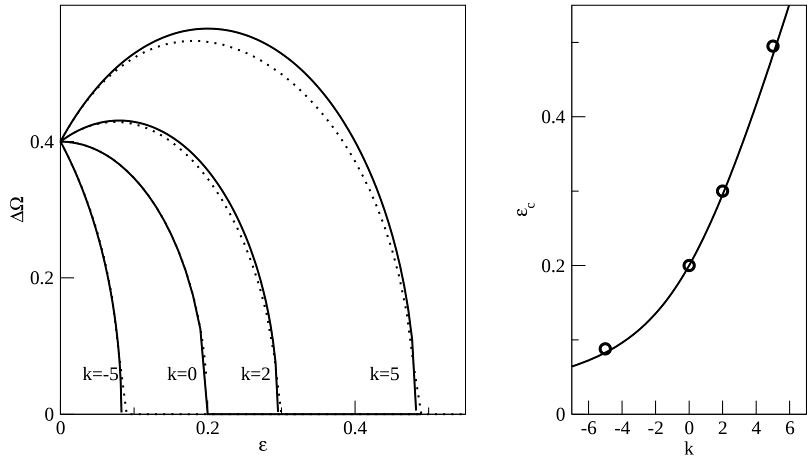

Figure 4. Left: Frequency difference of two coupled Landau-Stuart oscillators (15) with $\Delta q = k\Delta\omega$ as a function of coupling strength for different values of k (ω_1=1.2, ω_2=0.8). Plotted are the simulation results (dotted lines) and the analytical result from Eq. (20) (solid lines). Right: Synchronization threshold, ϵ_c, as a function of k. Solid line: analytical result from Eq. (20) by making $\Delta\Omega = 0$, circles: numerical results.

With this expression the transition to synchronization of two coupled nearly identical, weakly nonlinear oscillators has been described in the full coupling range. Thus, the transition to synchronization entirely depends the product $\Delta\omega\Delta q$. For $\Delta q = 0$ expression (20) reduces to the well known beat frequency of two coupled isochronous phase oscillators (11). In general however, $\Delta q \neq 0$ and we can write $\Delta q = k\Delta\omega$. Fig. (4) shows the results for different correlations between $\Delta\omega$ and Δq. We observe a good agreement of the analytical result (20) with the numerical simulations as long as both oscillators differ not too much. In particular, for positive values of k we find anomalous synchronization, whereas negative values of k lead to an enhancement of synchronization, as expected. Note that anomalous synchronization is effective not only at the onset of coupling but has important consequences also in the regime of larger coupling levels. This can be observed for example in the synchronization threshold ϵ_c which is shifted substantially by increasing levels of $|\Delta q|$.

To gain an intuitive understanding, assume that for small coupling levels, $\epsilon \ll \epsilon_c$, the oscillators (16) are rotating nearly independently. To take this into account we set $\sum \sin\phi \approx \sum \cos\phi \approx 0$ and the system transforms into a pair of independent oscillators

$$\begin{aligned} \dot{r}_i &= r_i(1 - r_i^2) - r_i\epsilon \\ \dot{\theta}_i &= \omega_i + q_i(1 - r_i^2). \end{aligned} \tag{21}$$

Thus, the average effect of small coupling is to reduce the radius of each limit cycle independently of its parameter values to the stable equilibrium $r_i^{*2} = 1 - \epsilon$. After relaxation to this radius we are left with the following equation for the phase

$$\dot{\theta}_i(\epsilon) = \omega_i + q_i\epsilon. \tag{22}$$

From expression (22) it is clear that in the range of small coupling the frequencies $\Omega_i(\epsilon)$ take a very simple form. The physical interpretation is straightforward. Due to the interaction the oscillators are perturbed off their limit cycle. On average this leads to a radial contraction of each limit cycle which produces a shift of the angular velocity proportional to the value of the shear term q_i.

Now recall the oscillators differ in their respective values of ω_i and q_i. Inserting $\Delta q = k\Delta\omega$ into (22) we retrieve formula (20) for the difference of observed frequencies up to first order in ϵ, i.e. $\Delta\Omega(\epsilon) = (1 + k\epsilon)\Delta\omega$. Thus the standard deviation is an increasing function of coupling strength when $k > 0$ and a decreasing function when $k < 0$. When $k = 0$ we are only varying the natural frequency. If the 'faster' oscillator (with higher natural frequency, ω_i) has a stronger shear of phase flow (higher value of q_i) compared to the 'slower' one, then small coupling leads to an enlargement of the frequency difference between the 'faster' and the 'slower' oscillator. Therefore, if the ω_i increase with q_i then small coupling tends to desynchronize the oscillators.

We want to stress that assuming different values of ω and q in both oscillators we always observe such anomalous synchronization effects, either inhibiting or enhancing synchronization. Since the Landau-Stuart model gives a generic description for weakly nonlinear oscillators of type (1) we expect that similar effects are always present in the synchronization transition of two non-identical oscillators which vary in both natural frequency and non-isochronicity. Thus, we expect for the observed frequencies of two coupled oscillators (1)

$$\Omega_i = \Omega_i(\chi_i, \epsilon) \approx \omega(\chi_i) + \epsilon q(\chi_i) + O(\epsilon^2). \tag{23}$$

In this case $q(\chi_i)$ describes the frequency response of each oscillator to the onset of interaction. Eq.(23) can exactly be derived from (3) in the limit $\epsilon \ll 1$, using a random phase approximation and averaging[26]

$$q(\chi_i) = \frac{1}{2\pi} \int_0^{2\pi} \Gamma_{ij}(\Delta\theta_{ji})\, d\Delta\theta_{ji}\,. \tag{24}$$

Using formula (5) then in principle the characteristics ω_i and q_i can be calculated from the basic equations (1).

With this theory of Eq.(23) we are now ready to explain the unusual transition to synchronization, which is exhibited by the two coupled foodweb models (Fig.3).[26] The fact that nonisochronicities must play in important role is immediately obvious from the fast drop of $\Omega_i(\epsilon)$ (compare to Eq.(22)). From such plots we can easily estimate the amount of nonisochronicity in a given model, even when the dynamcis are chaotic. Here, for example, we find that the nonisochronicity of the foodweb model is negative.

6. ASYMMETRIC COUPLING AND PERIODICALLY FORCED OSCILLATOR

In the previous sections we have discussed how anomalous effects can emerge when there is a correlation between two system characteristics such as nonisochronicity and natural frequency. Now we show that similar effects arise even when the oscillators have identical (or nearly identical) nonisochronicity, q, if the coupling between the oscillators is asymmetrical. Assume again two oscillators (15) which are now coupled with strength ϵ_1 and ϵ_2, respectively. In this case of asymmetrical coupling we find for the phase difference in a analogy to (18)

$$\dot{\phi} = \Delta\omega - (\epsilon_2 + \epsilon_1)\sin\phi - (\epsilon_2 - \epsilon_1) q(\cos\phi - 1). \tag{25}$$

Proceeding as in the previous section the observed frequency difference yields

$$\Delta\Omega(\epsilon) = \sqrt{\Delta\omega^2 + 2q\Delta\omega(\epsilon_2 - \epsilon_1) - (\epsilon_1 + \epsilon_2)^2}. \tag{26}$$

By comparison with (20) it is immediately evident that we find anomalous enlargement when the product $q\Delta\omega\Delta\epsilon > 0$ and anomalous synchronization enhancement if $q\Delta\omega\Delta\epsilon < 0$. Thus, if the oscillators are non-isochronous the asymmetry of coupling is reflected by an asymmetry of the locking regions.

Especially interesting is the limiting case of an externally forced oscillator (unidirectional coupling), which arises when we set $\epsilon_1 = 0$. In this case (25) goes over to

$$\dot{\phi} = \Delta\omega - \epsilon\,[\sin\phi + q(\cos\phi - 1)]. \tag{27}$$

Here, for simplicity we denote $\epsilon_2 = \epsilon$. This equation describes the evolution of the phase difference between a single nonisochronous oscillator and a periodically driving force, where the natural frequency of the oscillator and the driving frequency have a frequency mismatch of $\Delta\omega$.

Due to the stong impact of anomalous synchronization on the synchronization threshold also the geometry of the Arnold tongue will be modified in dependence of the nonisochronicity q. Recall that without nonisochronicity, $q = 0$, the border of the Arnold tongue is given by the two lines $\epsilon = |\Delta\omega|$ in the $(\Delta\omega, \epsilon)$-plane. However, if $q \neq 0$ the slope of the borderlines is changed. As can easily be demonstrated,[26] in the presence of nonisochronicity the Arnold tongue is simply rotated by an rotation angle α of

$$\tan(2\alpha) = q. \tag{28}$$

In the limit of infinit large nonisochronicity the Arnold tongue is rotated by 90^0 degrees. In other words the rotation angle of the Arnold tongue is a measurement for the nonisochronicity of oscillation. Thus, the whole synchronization transition depends on the sign of $\Delta\omega$. If $q > 0$ and the natural frequency of oscillation is larger than the driving frequency, then the synchronization threshold is enlarged. Otherwise, the synchronization threshold is reduced.

7. PHASE SYNCHRONIZATION UNDER PRESENCE OF NOISE

In the following we investigate how this transition to synchronization is modified under the presence of additive white noise of strength σ. In the nonisochronous case the results are well known.[3] Under the influence of uncorrelated noise the observed frequency difference $\Delta\Omega(\epsilon, \sigma)$ is still a monotonincally decreasing function of coupling strength ϵ but can never become zero due to noise induced phase slips. One natural measure, then is to investigate the distribution of phase differences.

Here now, we are interested in the consequences when the oscillators are characterised by nonzero value of nonisochronicity, $q_i \neq 0$. As a toy model we use again the foodweb model (14) for which the value of q_i can be estimated from the slope of $\Omega_i(\epsilon)$ in Fig.3. As has been worked out in the previous section, in the case of symmetrical coupling $\epsilon_1 = \epsilon_2$ the crucial parameter is the *difference* of nonisochronicities Δq. This value, even though present in our example ecological model is rather small. Therefore, here we simulate two unidirectionally coupled foodweb models ($\epsilon_1 = \epsilon$, $\epsilon_2 = 0$) so that we can expect to see nonisochronicity effects even though the difference Δq is rather small.

Fig.5 depicts the simulation results. Without coupling, $\epsilon = 0$, we find the usual result where the phase difference is linearly decreasing with time due to the natural frequency mismatch of both oscillators, $b_{1,2} = b_0 \pm \Delta b$. The only effect of the noise then is to give rise to small fluctuations of $\phi(t)$ around the linear decline. However, as shown in Fig.5, very unusal results appear when coupling is switched on, $\epsilon \neq 0$. For intermediate values of coupling strength, $\epsilon = 0.05$, the slope of the phase evolution becomes positive. As a consequence, the observed frequency difference $\Delta\Omega(\epsilon)$ becomes inverted. Only for very large coupling strength synchronization sets in again and the slope of $\phi(t)$ is zero. More systematically this is explored in the right hand side of Fig.5 where the observed frequency difference is plotted as a function of coupling strength. Cleary it can be observed that $\Delta\Omega(\epsilon, \sigma)$ changes sign for intermediate values of ϵ. Obviously there are two different locking regions, one for small coupling levels and a second for large coupling levels.

Fig.5 also includes a simulation with two identical oscillators, i.e. two foodweb models without parameter mismatch $\Delta b = 0$. In this case both oscillators of course have the same natural frequency $\Delta\omega = 0$ and therfore without noise they are always perfectly phase synchronized, indendent of the coupling strength. However, with the inclusion of noise this is not anymore the case. As is demonstrated in Fig.5 with the onset of coupling both oscillators are forced to rotate with nonidentical mean frequencies. The necessary breaking of symmetry, of course, stems from the unidirectionality of coupling. Thus, we find that in the presence of noise and nonisochronicity asymmetrical coupling is able to desynchronize to identical oscillators.

These results were obtained for the extreme case of unidirectional coupling. We have obtained very similar results for other non-symmetrical coupling schemes where both coupling strength are nonzero, $\epsilon_i \neq 0$. Further, we have found similar results for different types of oscillators with nonzero nonisochronicity.

These simulation results and especially the inversion of the observed frequency difference can be understood from our phase description of the previous section. Using (27) we can calculate the potential (8)

$$V(\phi) = -(\Delta\omega + \epsilon q)\phi - \epsilon\,(\cos\phi + q\sin\phi). \tag{29}$$

The phase difference of the two coupled oscillators can then be desribed as a noisy particle in the tilted periodic potential (29). Here, the increase of coupling has two different effects. First, it leads to a periodic modulation of the potential, $\epsilon\,(\cos\phi + q\sin\phi)$, which is responsible for the phase locking. On the other hand however, coupling leads to an overall tilt of the potential with slope $\Delta\omega + \epsilon q$. This tilting of the potential favours a phase drift. The mean slope of the potential depends on the balance of $\Delta\omega$ and ϵq. Therefore, this second effect is only possible when $q \neq 0$, i.e. when the oscillators have nonvanishing nonisochronicity.

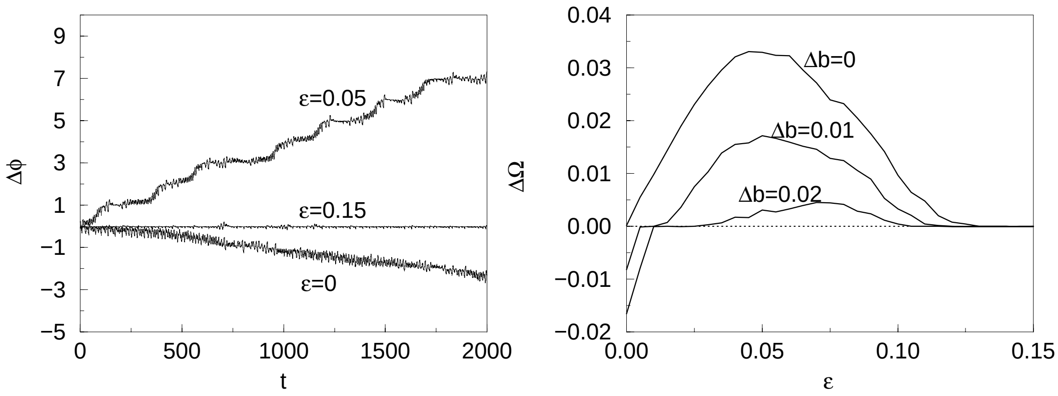

Figure 5. Transition to synchronization in two unidirectional coupled foodweb models (14) in the presence of additive white noise. Left) Phase evolution, $\phi(t)$ for different levels of coupling strength $\epsilon = 0$, $\epsilon = 0.05$, $\epsilon = 015$ and $\sigma = 0.2$. Right) Frequency difference $\Delta\Omega(\epsilon)$ as a function of coupling strength. Parameter mismatch between both oscillators $b_{1,2} = 0.98 \pm \Delta b$ with $\Delta b = 0.01$ (left and right) and $\Delta b = 0.02$, $\Delta b = 0$ (right). Parameters otherwise as in Fig.3.

The simulation results of Fig.5 can now be explained as follows. Without coupling strength the potential is tilted to the left and there is no periodic modulation. Therefore the phase difference is decreasing with time. When small coupling is switched on, local minima are generated in the potential which leads to a locking of the phase difference. This is the first locking region, $\Delta\Omega = 0$, for small coupling levels. When coupling strength is increased even more, the tilt of the potential can become so large that the particle is moving to the right and therefore the mean observed frequency becomes positive. Note, that this effect in principle depends on the effect of noise because the particle needs to jump of the potential barriers. Only for very large coupling strength locking sets in again because the potential barriers are increasing more and more.

8. CONCLUSION

Here we have shown that the synchronization of two oscillators with nonvanishing nonisochronicites is intricate, especially under the presence of additive noise. We have demonstrated that the usual picture of the transition to synchronization can then be totally modified and 'anomalous effects' may arise where coupling can increase or even invert the frequency disorder. It would be interesting to extend these studies to systems with a large number of oscillators. Another interesting objective would be to investigate the transition to synchronization when the coupling strength is a function of time. For example one can allow for noisy or periodic changes of the coupling strength. The synchronization is then determined in terms of a noisy particle in a rocked tilted periodic potential.

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